Vivien Chen (NTHU)

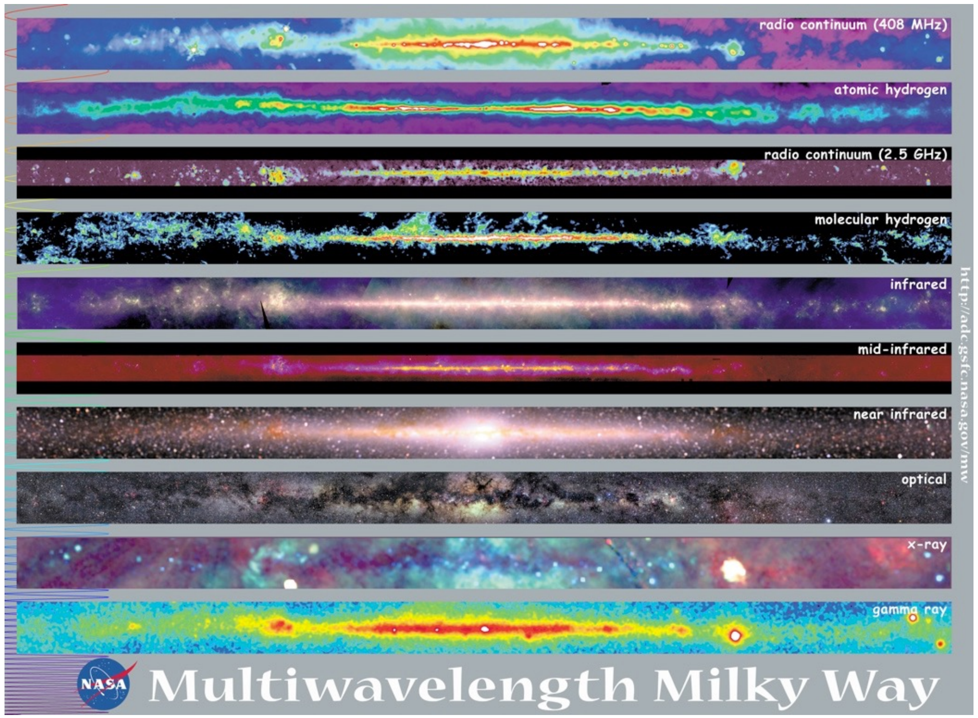
Aperture Synthesis and Interferometers

Recommended readings:

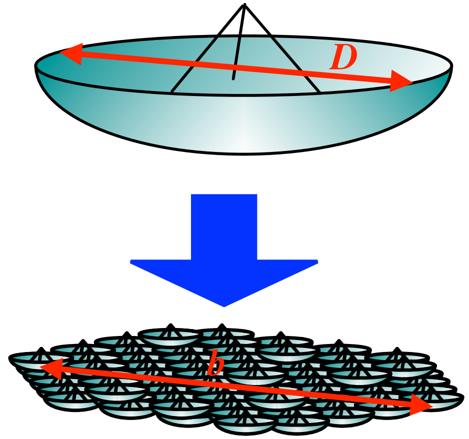
Clark 1999, ASPC, 180, 1; Thompson 1999, ASPC, 180, 11

Fundamentals of Radio Interferometry

Observation strategies

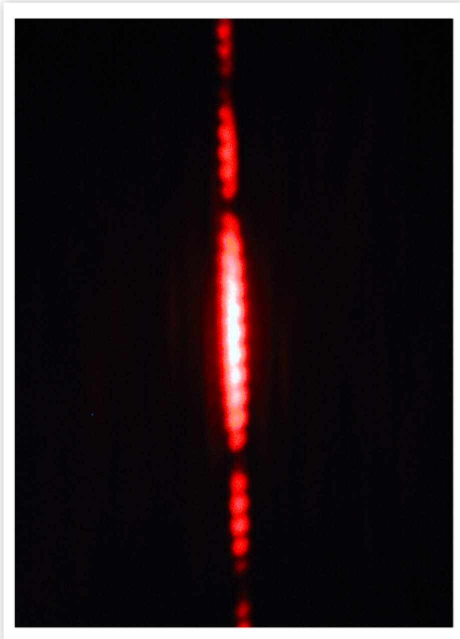


What are interferometers?



$$\theta = 1.22\frac{\lambda}{D}$$

$$\theta = 1.22\frac{\lambda}{b}$$



Introduction to Interferometry

The double-slit experiment as an example

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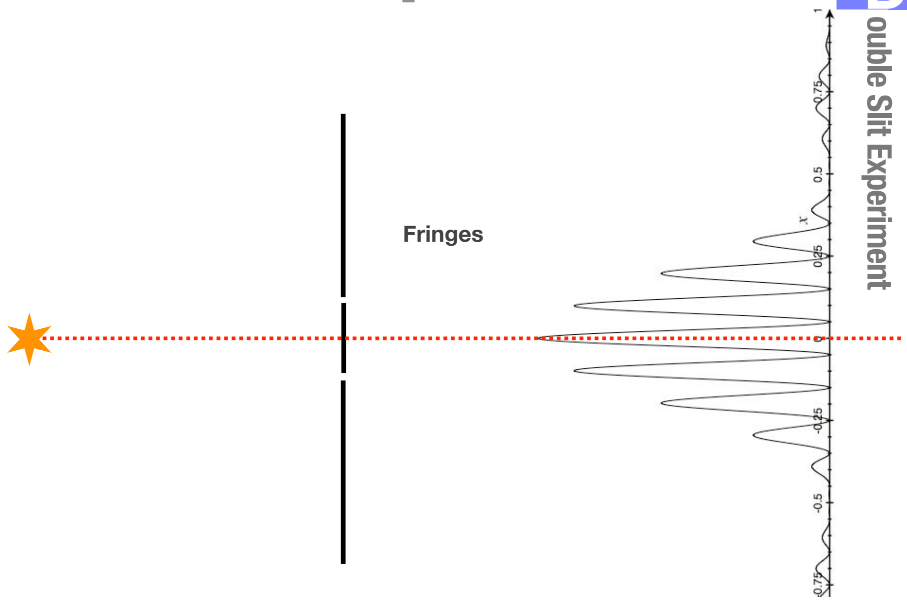
Double-slit Experiment



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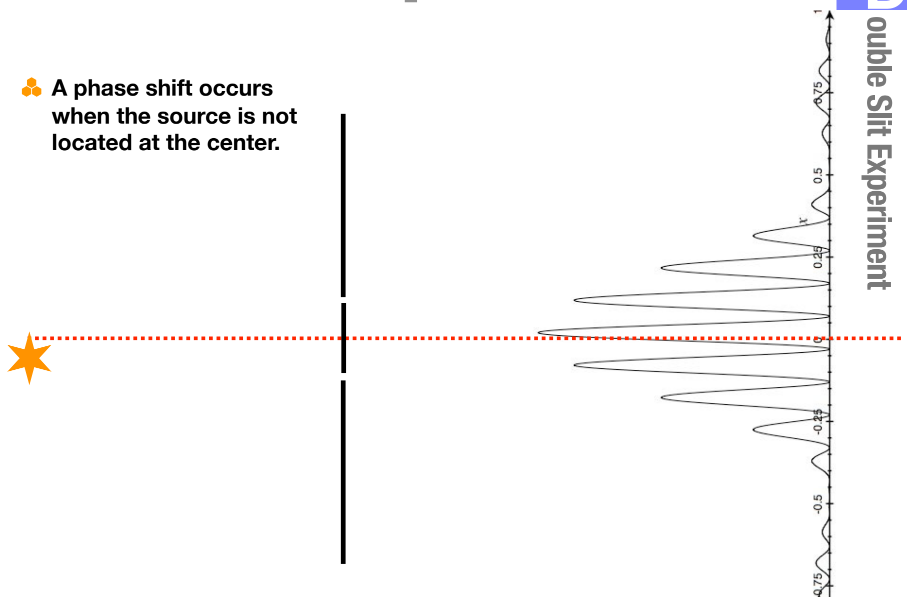
Double-slit Experiment

- A phase shift occurs when the source is not located at the center.



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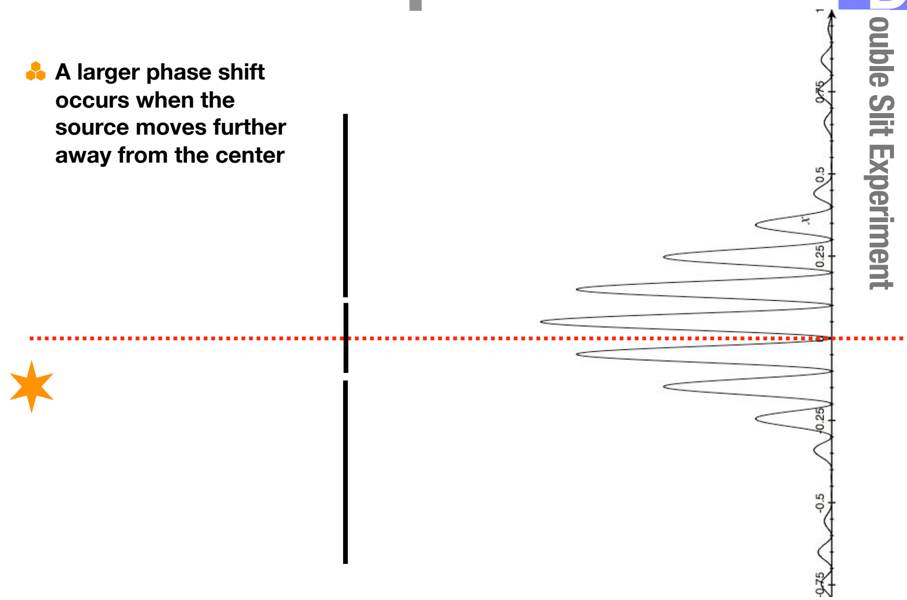
Double-slit Experiment

- A larger phase shift occurs when the source moves further away from the center



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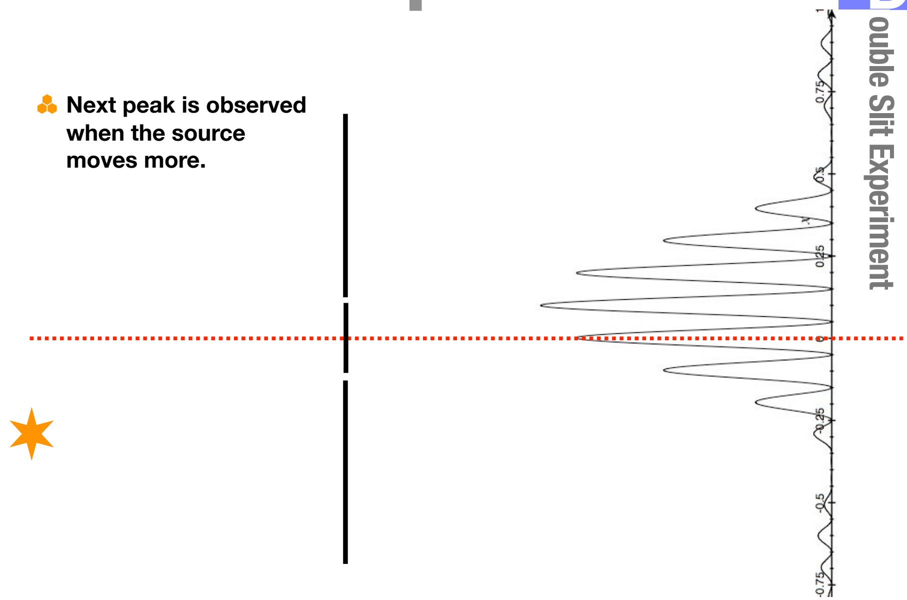
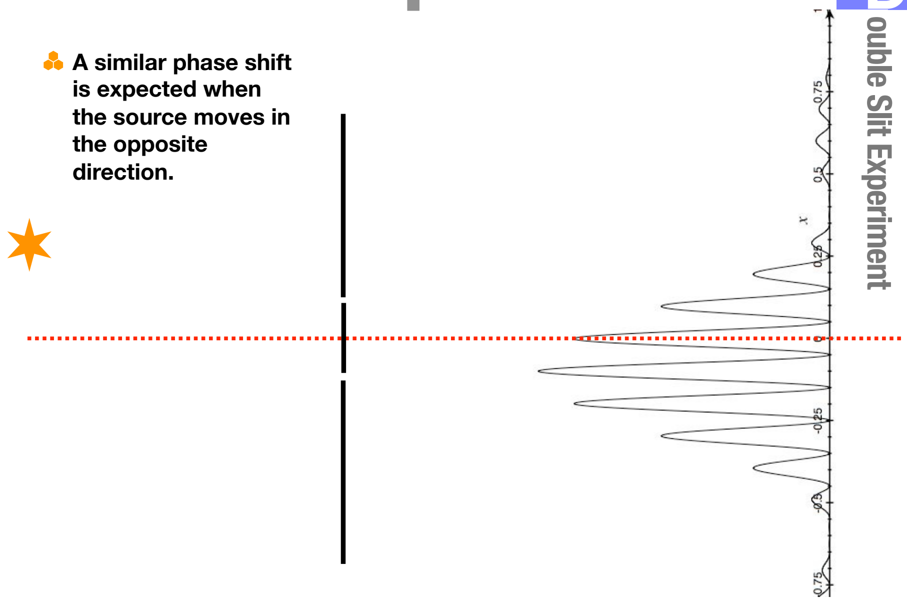
Double-slit Experiment

- Next peak is observed when the source moves more.



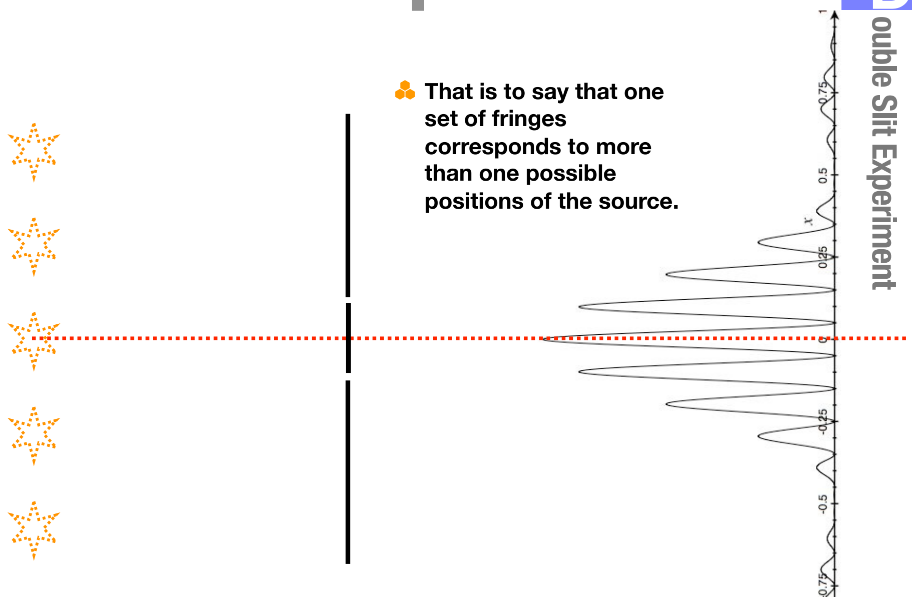
Double-slit Experiment

- A similar phase shift is expected when the source moves in the opposite direction.



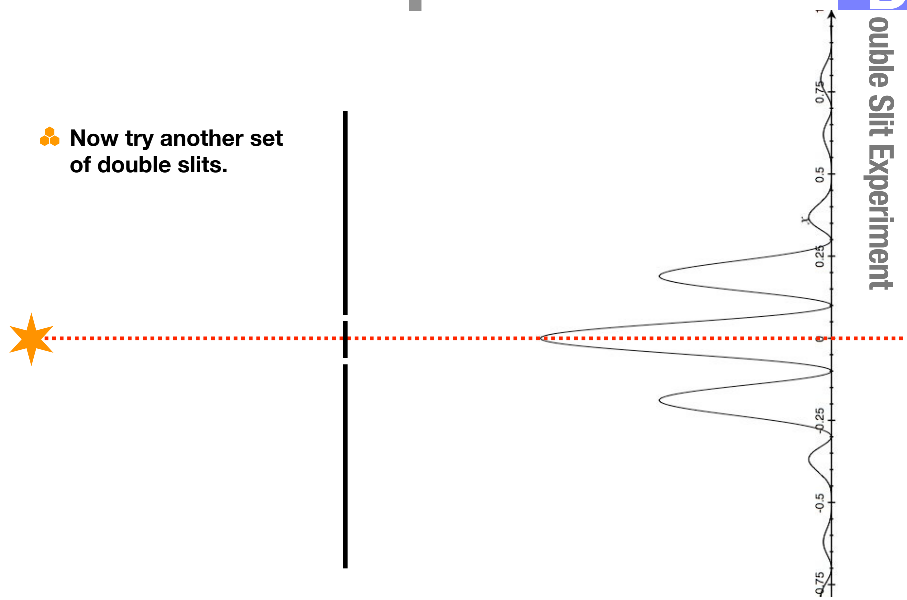
Double-slit Experiment

- That is to say that one set of fringes corresponds to more than one possible positions of the source.



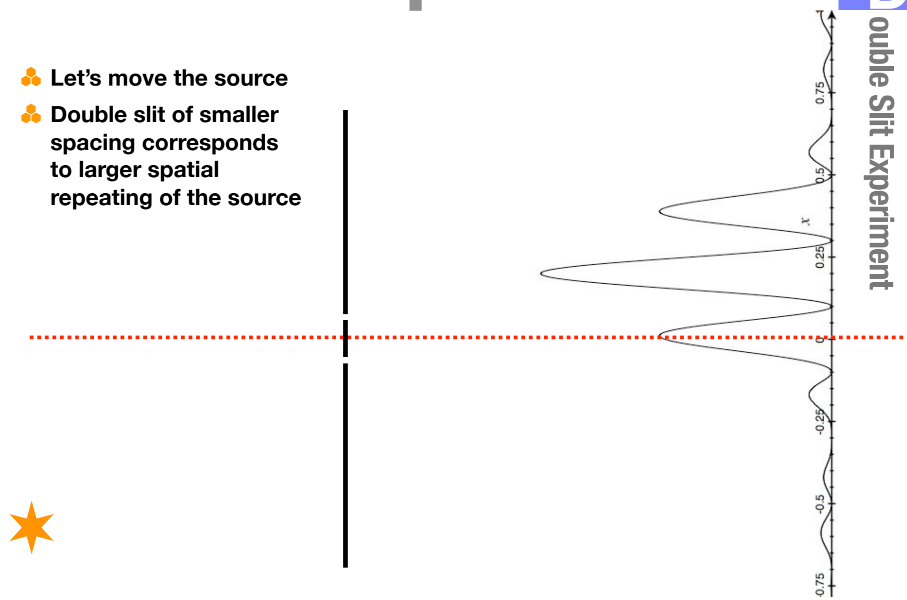
Double-slit Experiment

- Now try another set of double slits.



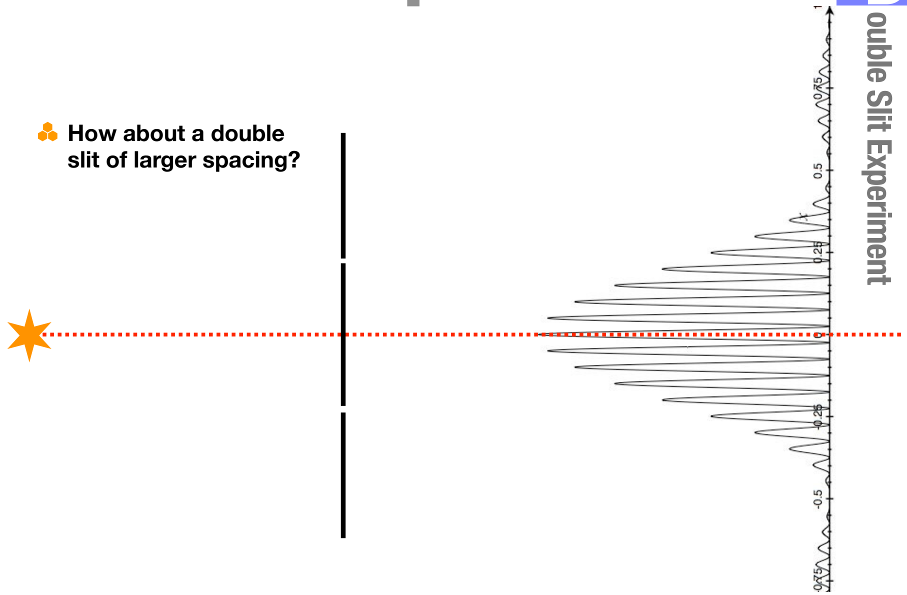
Double-slit Experiment

- Let's move the source
- Double slit of smaller spacing corresponds to larger spatial repeating of the source



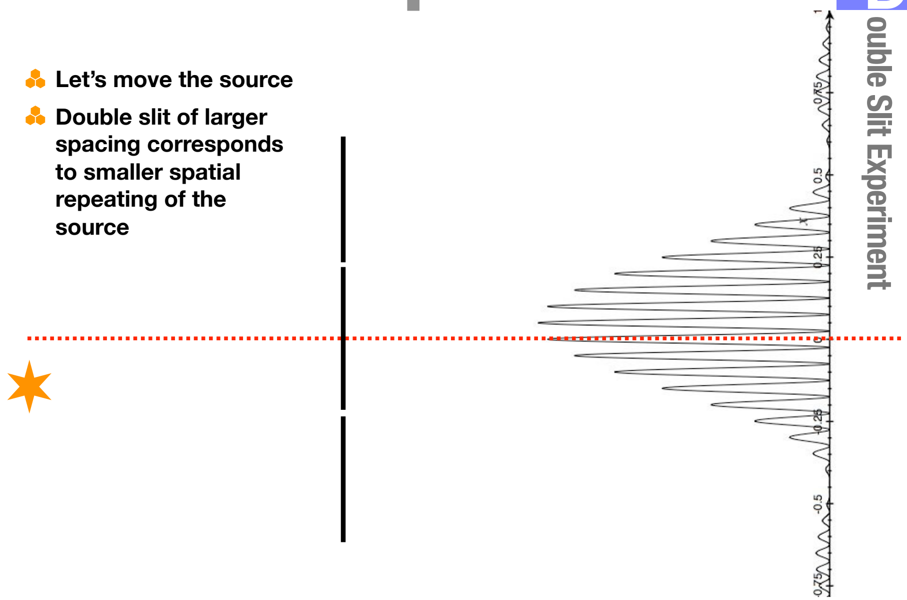
Double-slit Experiment

- How about a double slit of larger spacing?



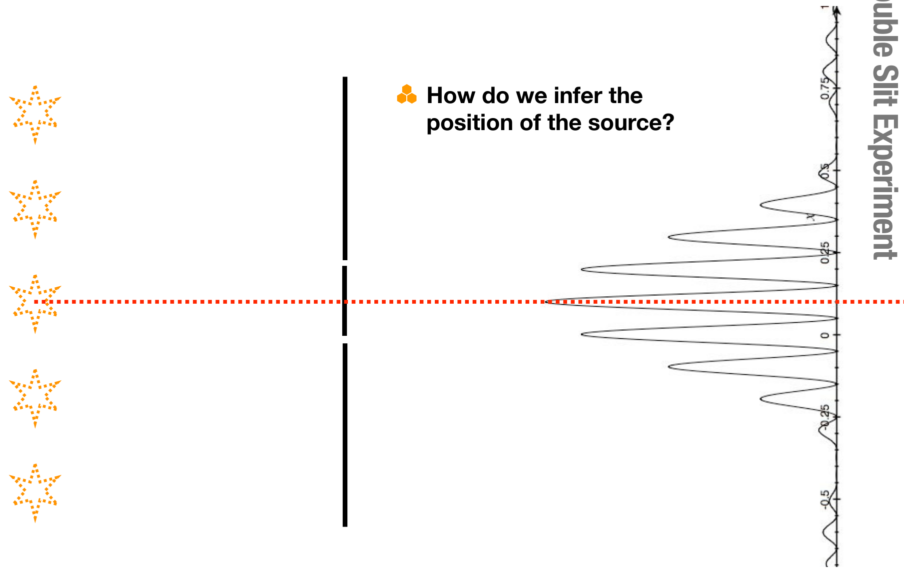
Double-slit Experiment

- Let's move the source
- Double slit of larger spacing corresponds to smaller spatial repeating of the source



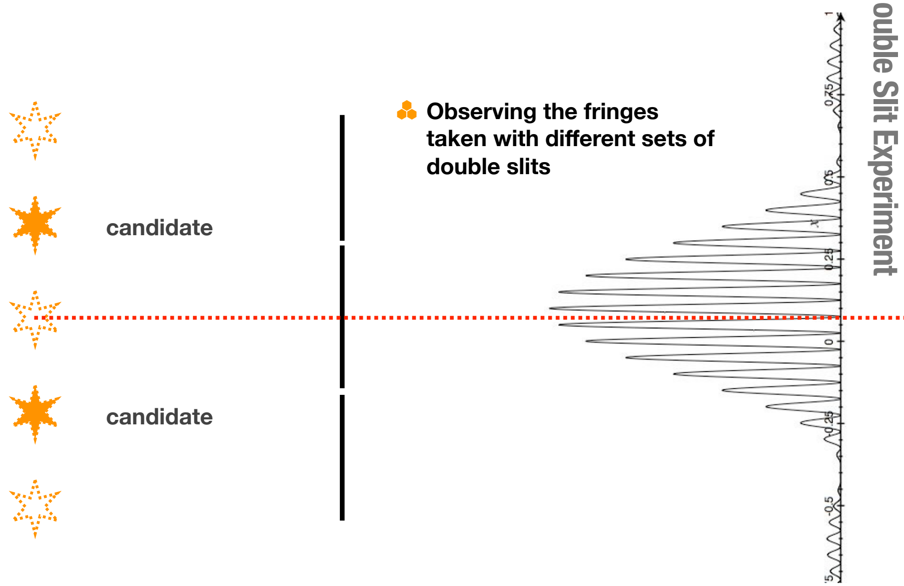
Double-slit Experiment

- How do we infer the position of the source?



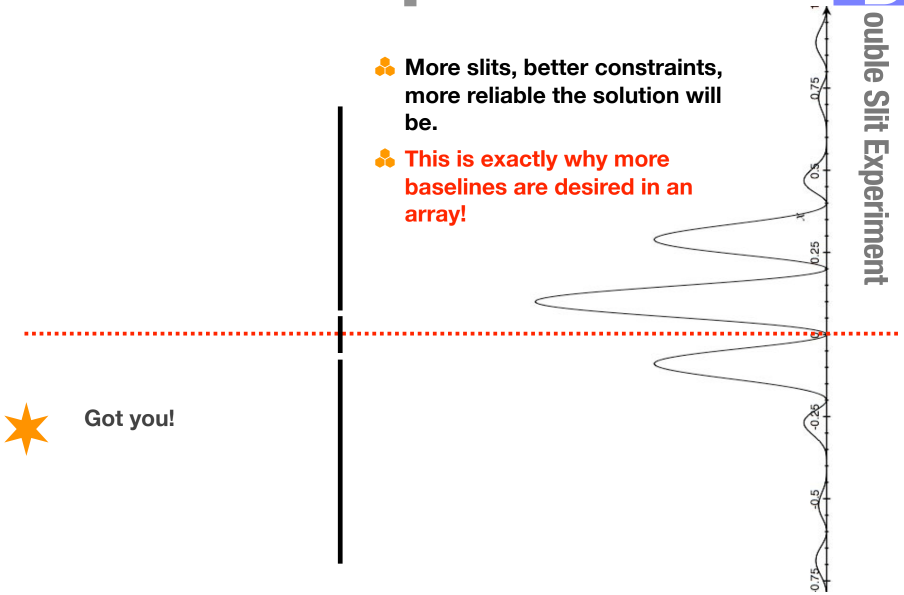
Double-slit Experiment

- Observing the fringes taken with different sets of double slits



Double-slit Experiment

- More slits, better constraints, more reliable the solution will be.
- This is exactly why more baselines are desired in an array!

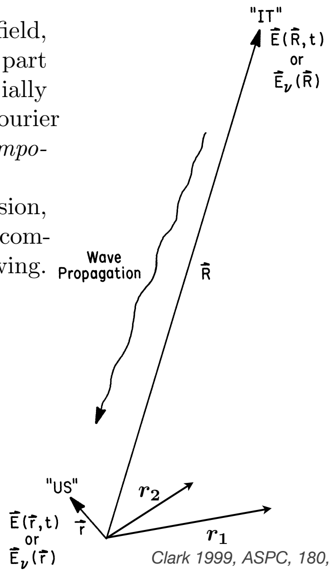


Got you!

Observed Electric Field

Consider a finite time interval of a varying field, whose magnitude may be expressed as the real part of the sum of the Fourier series with exponentially time-varying functions. The coefficients of this Fourier series, $\boldsymbol{E}_\nu(\boldsymbol{R})$, are the *quasi-monochromatic components* of the electric field, $\boldsymbol{E}(\boldsymbol{R}, t)$.
To avoid unnecessary complications of the discussion, let's consider only a single quasi-monochromatic component and introduce five assumptions as following.

Scalar field assumption. Ignore the fact that EM wave is a vector phenomenon, and treat it as if it were simply a scalar field.
Projection on a celestial sphere. Project all the emitting phenomena on a celestial sphere without describing the structure of the emitting regions in the their dimension.



Clark 1999, ASPC, 180, 1

Spatial Coherence Function I

Propagation through the vacuum. The space within the celestial sphere is empty, and Huygens' Principle can be applied

$$E_\nu(\boldsymbol{r}) = \int E_\nu(\boldsymbol{R}) \frac{e^{2\pi i\nu|\boldsymbol{R}-\boldsymbol{r}|/c}}{|\boldsymbol{R}-\boldsymbol{r}|} \mathrm{d}S,$$

where $\mathrm{d}S$ is the element of surface area on the celestial sphere. The correlation of the field at points $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ is defined as

$$\begin{aligned} V_\nu(\boldsymbol{r_1}, \boldsymbol{r_2}) &\equiv \langle E_\nu(\boldsymbol{r_1}) E_\nu^*(\boldsymbol{r_2})\rangle \\ &= \left\langle \int\int E_\nu(\boldsymbol{R_1}) E_\nu^*(\boldsymbol{R_2}) \frac{e^{2\pi i\nu|\boldsymbol{R_1}-\boldsymbol{r_1}|/c}}{|\boldsymbol{R_1}-\boldsymbol{r_1}|} \frac{e^{2\pi i\nu|\boldsymbol{R_2}-\boldsymbol{r_2}|/c}}{|\boldsymbol{R_2}-\boldsymbol{r_2}|} \mathrm{d}S_1 \mathrm{d}S_2 \right\rangle \end{aligned}$$

Spatially incoherent emission. Assuming that the radiation from sources is not spatially coherent, i.e. $\langle E_\nu(\boldsymbol{R_1}) E_\nu^*(\boldsymbol{R_2})\rangle = 0$ for $\boldsymbol{R_1} \neq \boldsymbol{R_2}$, we obtain

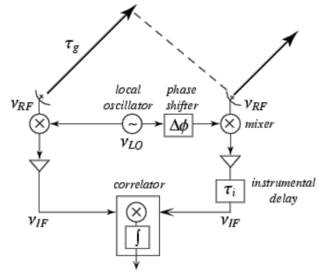
$$V_\nu(\boldsymbol{r_1}, \boldsymbol{r_2}) = \int \langle |E_\nu(\boldsymbol{R})|^2\rangle \frac{e^{2\pi i\nu|\boldsymbol{R}-\boldsymbol{r_1}|/c}}{|\boldsymbol{R}-\boldsymbol{r_1}|} \frac{e^{2\pi i\nu|\boldsymbol{R}-\boldsymbol{r_2}|/c}}{|\boldsymbol{R}-\boldsymbol{r_2}|} \mathrm{d}S.$$

Spatial Coherence Function II

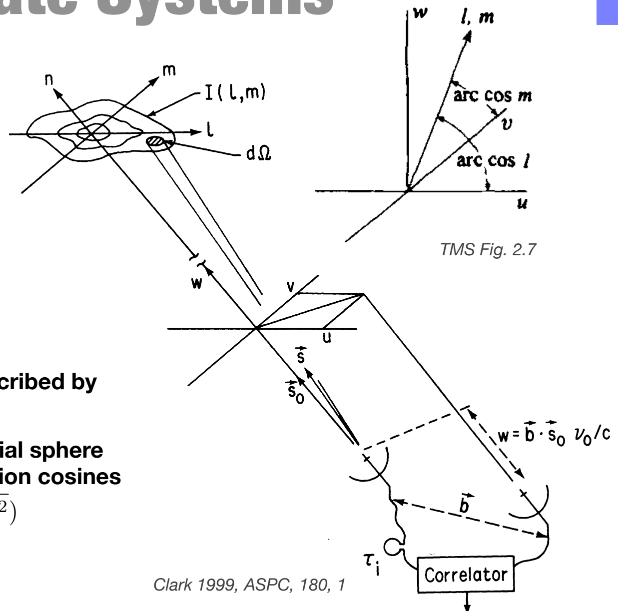
Substituting $\boldsymbol{s}$ for the unit vector $\boldsymbol{R}/|\boldsymbol{R}|$, $I_\nu(\boldsymbol{s})$ for the observed intensity $c\langle|E_\nu(\boldsymbol{s})|^2\rangle/4\pi$, $\mathrm{d}\Omega$ for the solid angle $\mathrm{d}S/|\boldsymbol{R}|^2$, and neglecting all terms with $|\boldsymbol{r}/\boldsymbol{R}|$, we obtain the spatial coherence function

$$\boxed{V_\nu(\boldsymbol{r_1}, \boldsymbol{r_2}) \simeq \int I_\nu(\boldsymbol{s})\, e^{-2\pi i\nu \boldsymbol{s}\cdot(\boldsymbol{r_1}-\boldsymbol{r_2})/c} \mathrm{d}\Omega.}$$

An interferometer is a device for measuring the spatial coherence function. The intensity distribution of the radiation as a function of direction $\boldsymbol{s}$ can be deduced in certain cases by measuring the spatial coherence function V as a function of $\boldsymbol{r_1} - \boldsymbol{r_2}$ and performing the inversion.
Further simplification involves our fifth and final assumption, which can be argued with two special cases of great interest.



Coordinate Systems



TMS Fig. 2.7

Clark 1999, ASPC, 180, 1

- **Measurements described by** $\lambda(u, v, w)$
- **Radiation on celestial sphere described by direction cosines** $(l, m, \sqrt{1-l^2-m^2})$

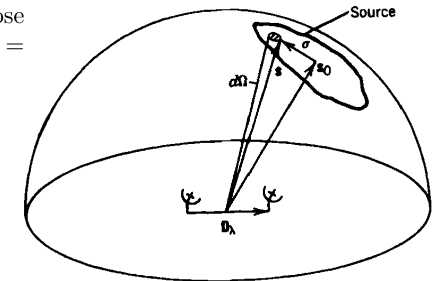
Spatial Coherence Function III

Given the direction cosines, we choose $\boldsymbol{s} = (l, m, \sqrt{1-l^2-m^2})$ and $\boldsymbol{s_0} = (0, 0, 1)$ so that

$$\frac{\nu \boldsymbol{s} \cdot (\boldsymbol{r_1} - \boldsymbol{r_2})}{c} = ul + vm + wn,$$

$$\frac{\nu \boldsymbol{s_0} \cdot (\boldsymbol{r_1} - \boldsymbol{r_2})}{c} = w$$

$$\mathrm{d}\Omega = \frac{\mathrm{d}l\,\mathrm{d}m}{n} = \frac{\mathrm{d}l\,\mathrm{d}m}{\sqrt{1-l^2-m^2}}.$$



TMS Fig. 3.1

Substituting the above relations, we find the spatial coherence function to be

$$V_\nu(u, v, w) = \int\int I(l, m)\, e^{-2\pi i[ul+vm+w(\sqrt{1-l^2-m^2}-1)]}\, \frac{\mathrm{d}l\,\mathrm{d}m}{\sqrt{1-l^2-m^2}},$$

where the integral is taken to be zero for $l^2 + m^2 \geq 1$.

Spatial Coherence Function IV

Coplanar Arrays. The first special case consider making all the measurements in a plane, i.e. $\boldsymbol{r_1} - \boldsymbol{r_2} = \lambda(u, v, w = 0)$. The spatial coherence function will take the form

$$V_\nu(u, v, w = 0) = \int\int I_\nu(l, m)\, \frac{e^{-2\pi i(ul+vm)}}{\sqrt{1-l^2-m^2}}\, \mathrm{d}l\,\mathrm{d}m.$$

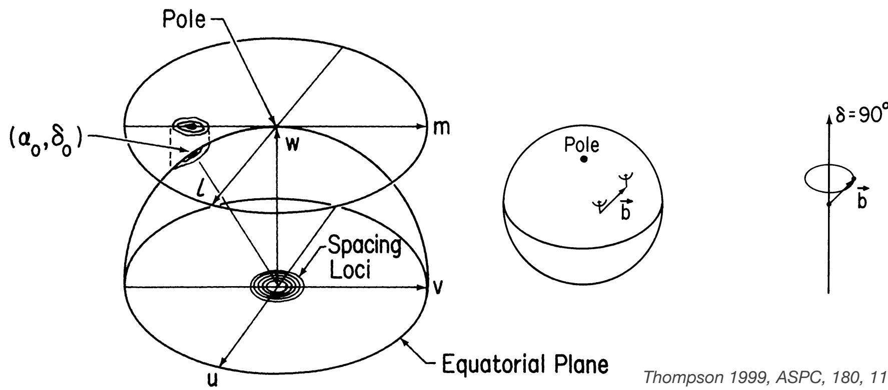
Sources in a small patch of sky. The second special case consider all the radiation of interest comes from only a small portion of the celestial sphere, i.e. $\boldsymbol{s} = \boldsymbol{s_0} + \boldsymbol{\sigma}$ with $\boldsymbol{s_0} \cdot \boldsymbol{\sigma} = 0$. In other words, $|l|$ and $|m|$ are small that $(\sqrt{1-l^2-m^2} - 1)w \simeq 0$ and the spatial coherence function becomes

$$V_\nu(u, v) = \int\int I_\nu(l, m) e^{-2\pi i(ul+vm)}\, \mathrm{d}l\,\mathrm{d}m,$$

where $V_\nu(u, v)$ is the coherence function relative to the **phase tracking center**, $\boldsymbol{s_0}$.

Coplanar Arrays

- **Coplanar array can be formed by placing all baselines along an East-West line, which zeros the components of the baseline vector parallel to the Earth's axis**
- **Distortion occurs at low elevation**

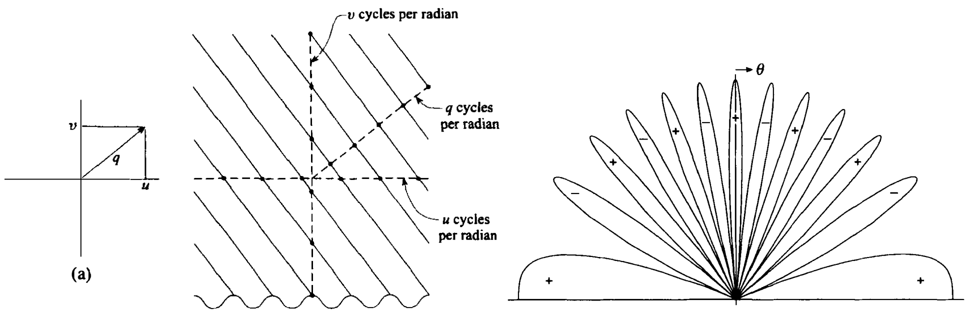


Thompson 1999, ASPC, 180, 11

Response of Interferometers

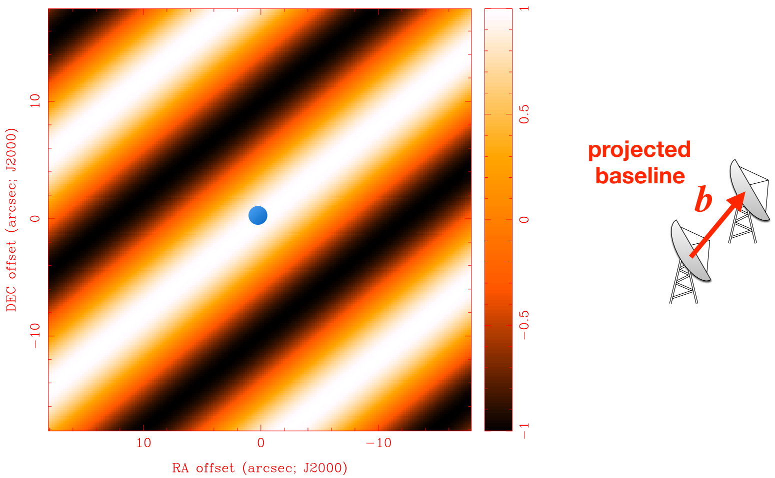
The complex visibility of the source is defined as

$$V_\nu(u,v) = \int\int I_\nu(l,m) e^{-2\pi i(ul+vm)}\,\mathrm{d}l\,\mathrm{d}m$$



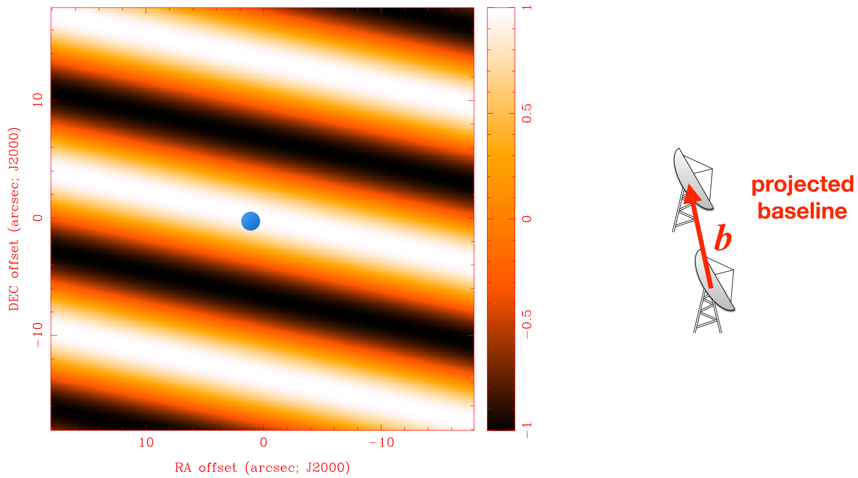
Visibility

$$V_\nu(u,v) = \int\int I_\nu(l,m) e^{-2\pi i(ul+vm)}\,\mathrm{d}l\,\mathrm{d}m$$



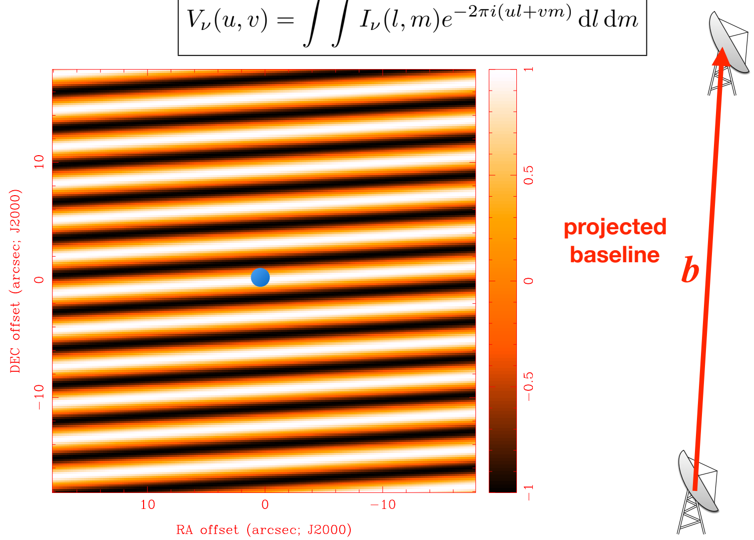
Visibility

$$V_\nu(u,v) = \int\int I_\nu(l,m) e^{-2\pi i(ul+vm)}\,\mathrm{d}l\,\mathrm{d}m$$



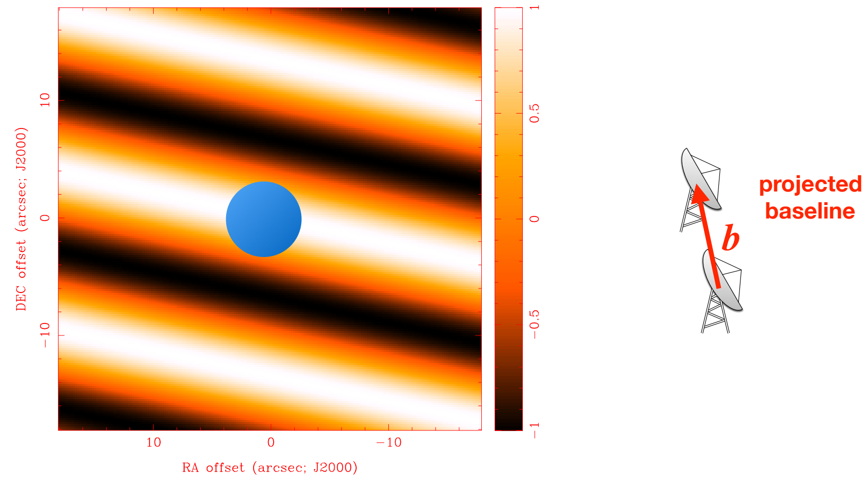
Visibility

$$V_\nu(u,v) = \int\int I_\nu(l,m) e^{-2\pi i(ul+vm)}\,\mathrm{d}l\,\mathrm{d}m$$



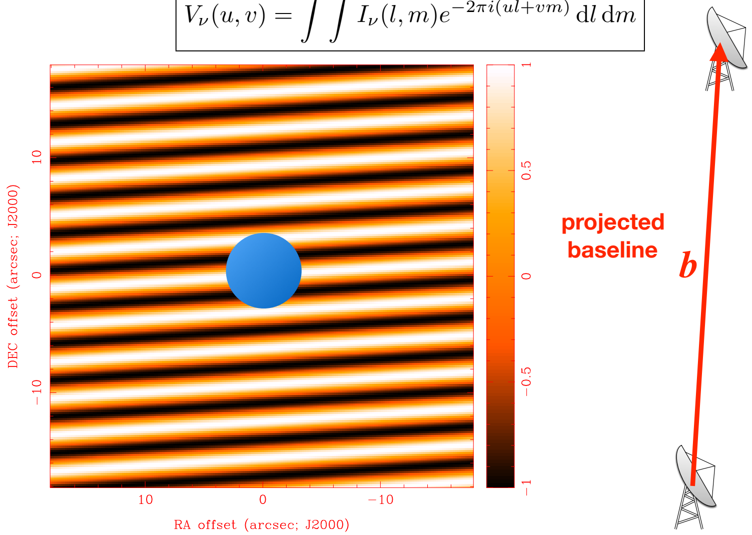
Visibility

$$V_\nu(u,v) = \int\int I_\nu(l,m) e^{-2\pi i(ul+vm)}\,\mathrm{d}l\,\mathrm{d}m$$



Visibility - Resolved Out

$$V_\nu(u,v) = \int\int I_\nu(l,m) e^{-2\pi i(ul+vm)}\,\mathrm{d}l\,\mathrm{d}m$$



Effect of Discrete Sampling

Given the above relationship between $V_\nu(u,v)$ and $I_\nu(l,m)$, it is obvious that the direct inversion reads

$$I_\nu(l,m) = \int\int V_\nu(u,v)\, e^{2\pi i(ul+vm)}\,\mathrm{d}u\,\mathrm{d}v.$$

In practice, V_ν is not known everywhere but is sampled at particular places on the $u-v$ plane described by a **sampling function**, $S(u,v)$, that $S(u,v) = 0$ where no data have been taken. One can compute

$$I_\nu^D(l,m) = \int\int V_\nu(u,v) S(u,v)\, e^{2\pi i(ul+vm)}\mathrm{d}u\,\mathrm{d}v,$$

where $I_\nu^D(l,m)$ is referred to as the **dirty image**; its relation to the ideal intensity distribution is

$$I_\nu^D = I_\nu \otimes B,$$

where $B(l,m)$ is the so-called **synthesized beam** or point spread function

$$B(l,m) = \int\int S(u,v)\, e^{2\pi i(ul+vm)}\,\mathrm{d}u\,\mathrm{d}v.$$

Effect of Primary Beam

In practice, the interferometer elements are not point probes which sense the voltage at that point, but are elements of finite size and directional sensitivity. The normalized reception pattern of each element, i.e. the **primary beam** needs to be included

$$V_\nu(u,v) = \int\int \mathcal{A}_\nu(l,m) I_\nu(l,m)\, e^{-2\pi i(ul+vm)}\,\mathrm{d}l\,\mathrm{d}m,$$

where $\mathcal{A}_\nu(l,m) = A_\nu(l,m)/A_{\nu,0}$.

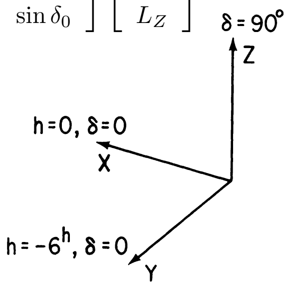
The calibration with the element directional sensitivity $\mathcal{A}_\nu$ should be postpoined to the final step of deriving the sky intensity distribution and it should simply divide the derived intensities. Such division is often referred to as **primary beam correction** and will, however, not only produce a better estimate of the actual intensities in this direction, but will also increase the errors in directions far from the phase tracking center.

Projection of Baselines I

With multi-element arrays, it is convenient to specify the antenna positions in a Cartesian coordinate system. For example, a system with X the direction of the meridian at the celestial equation, Y the East, and Z toward the North celestial pole. Let L_X, L_Y, and L_Z the corresponding coordinate differences for two antennas, the baseline components (u, v, w) are given by

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix} \sin H_0 & \cos H_0 & 0 \\ -\sin\delta_0 \cos H_0 & \sin\delta_0 \sin H_0 & \cos\delta_0 \\ \cos\delta_0 \cos H_0 & -\cos\delta_0 \sin H_0 & \sin\delta_0 \end{bmatrix} \begin{bmatrix} L_X \\ L_Y \\ L_Z \end{bmatrix},$$

where H_0 and δ_0 are the hour angle and declination of the phase reference position. The elements in the transformation matrix are the direction cosines of the (u, v, w) axes relative to (X, Y, Z) axes.



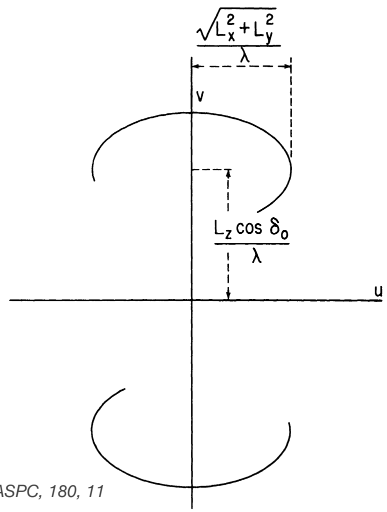
Thompson 1999, ASPC, 180, 11

Projection of Baselines II

Eliminating H_0 from the expressions for u and v, we can obtain the equation of an ellipse in the (u, v) plane:

$$u^2 + \left(\frac{v - (L_z/\lambda)\cos\delta_0}{\sin\delta_0}\right)^2 = \frac{L_X^2 + L_Y^2}{\lambda^2}.$$

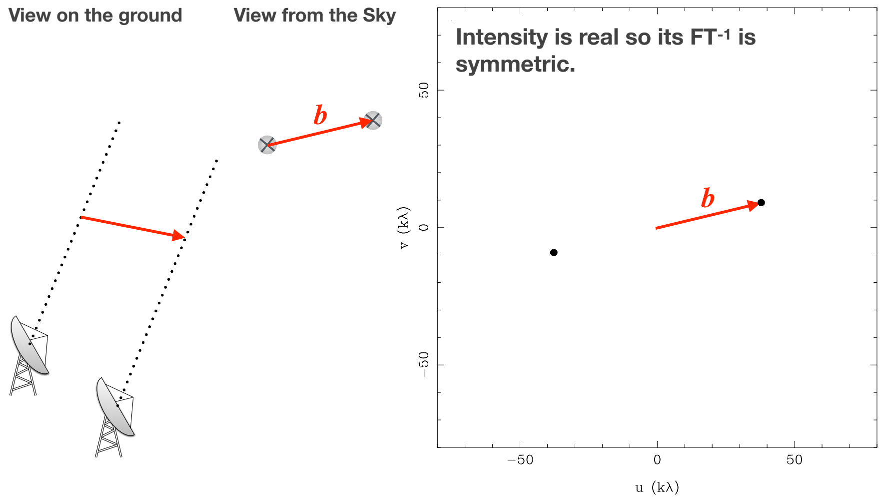
The ellipse is simply the projection onto the (u, v) plane of the circular locus traced out by the tip of the baseline vector. Since $I(l, m)$ is real, $V(-u, -v) = V^*(u, v)$. For an array of antennas, the ensemble of elliptical loci is the **sampling function**, $S(u, v)$.



Thompson 1999, ASPC, 180, 11

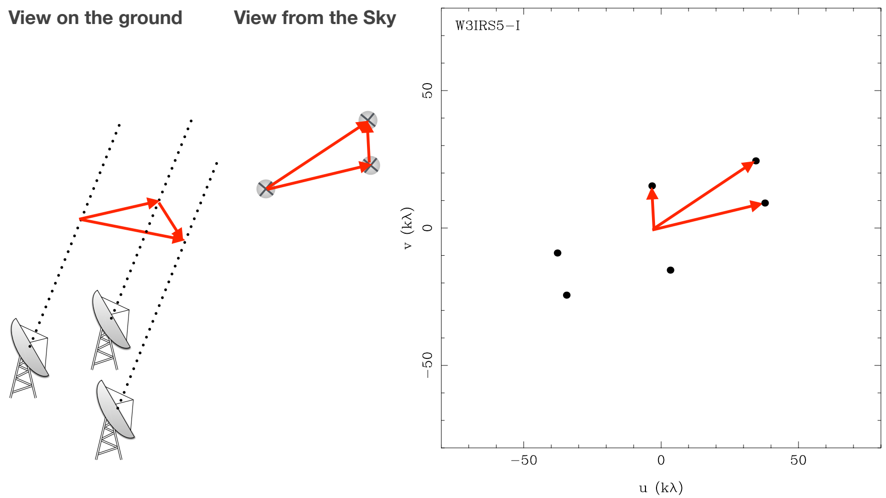
The u-v Plane

2 antennas, 1 sample



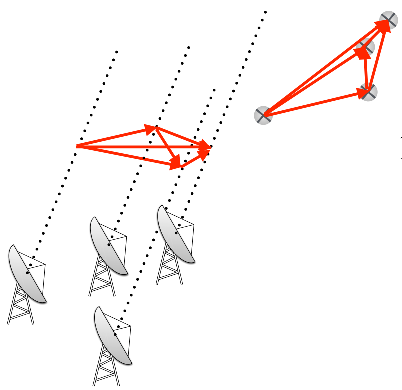
The u-v Plane

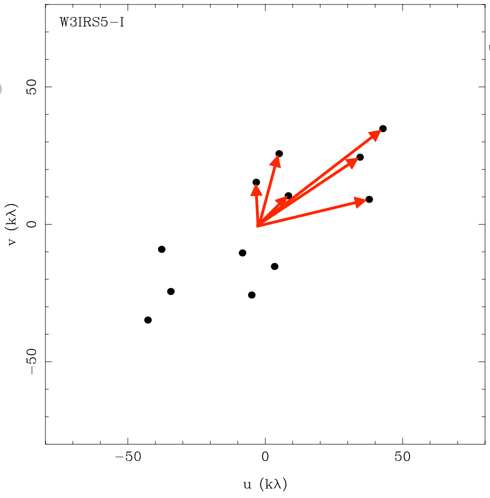
3 antennas, 1 sample



The u-v Plane

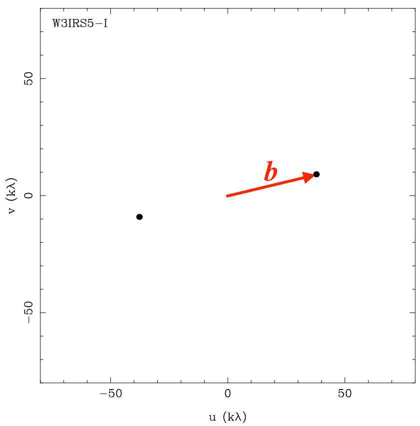
4 antennas, 1 sample

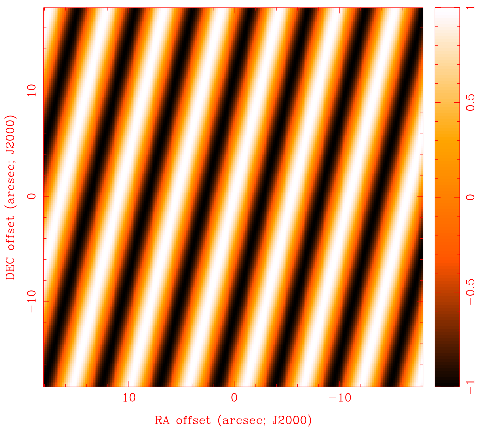




Aperture Synthesis

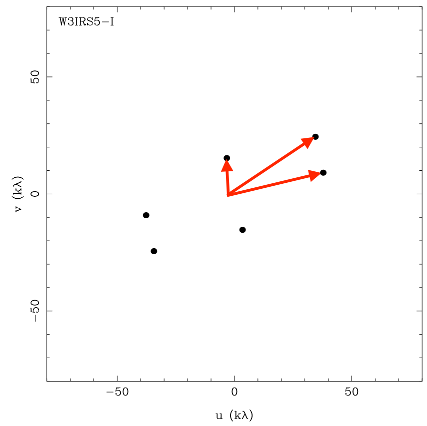
2 antennas, 1 sample

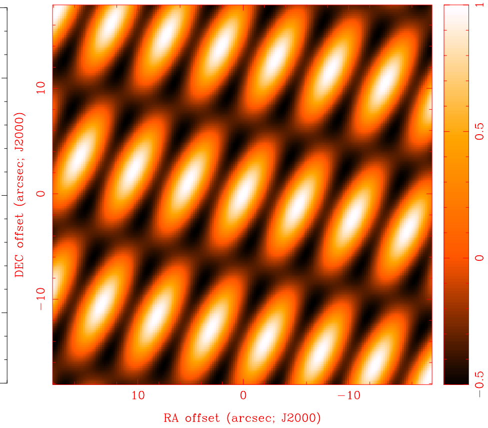




Aperture Synthesis

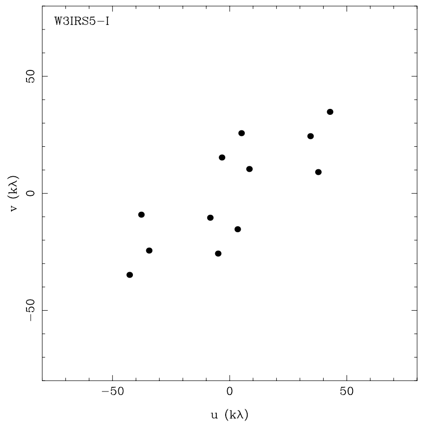
3 antennas, 1 sample

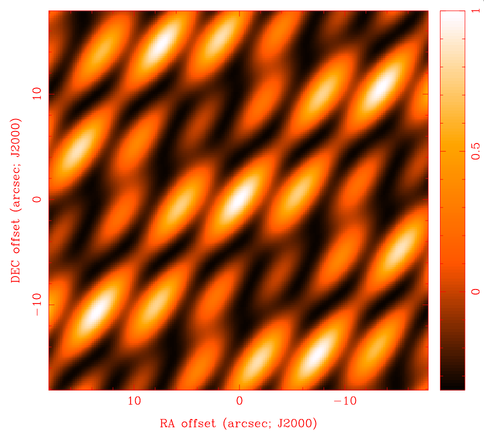




Aperture Synthesis

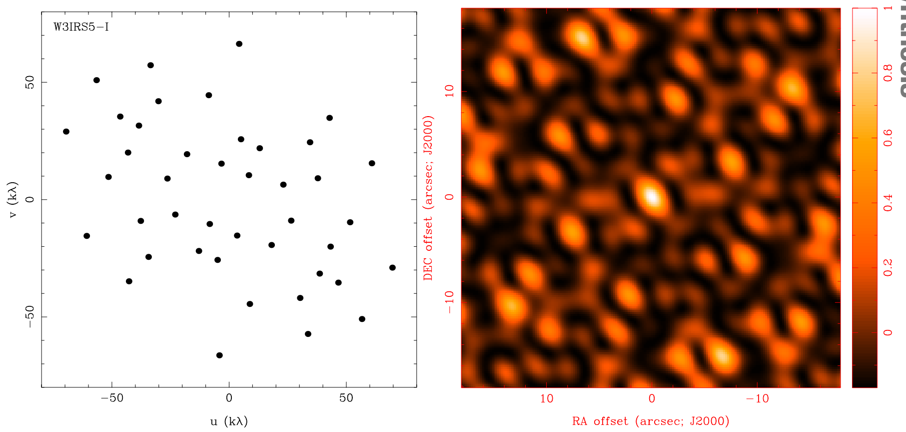
4 antennas, 1 sample





Aperture Synthesis

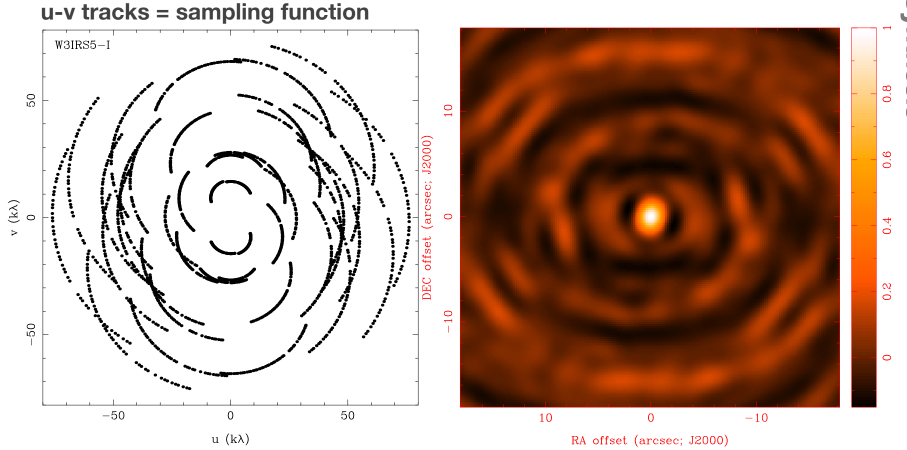
7 antennas, 1 sample



Aperture Synthesis

7 antennas, full track

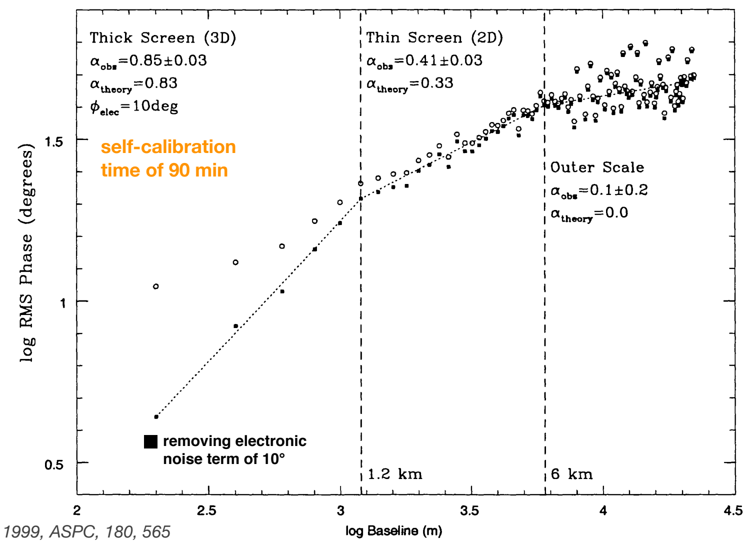
u-v tracks = sampling function



Observation Strategy with Interferometers

Tropospheric Phase Fluctuation

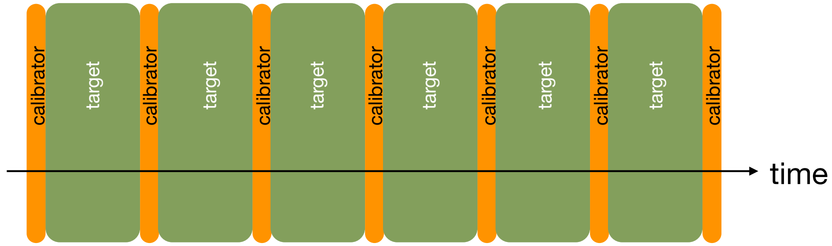
22 GHz Root Phase Structure Function (5400 sec) -- Jan. 27, 1997, 13mm



Carilli et al. 1999, ASPC, 180, 565

Basic Calibration Types

- **Bandpass calibration: correct frequency response by observing a bright source of featureless spectrum**
- **Flux calibration: correct visibility amplitudes by observing a source of known flux density**
- **Gain calibration: correct temporal phase fluctuation by repeatedly observing a calibrator of known structures to track what the troposphere is doing**



Gain Calibration I

Since the visibility is sampled at discrete times for each antenna pair, the array synthesis formulation is often written as

$$V_{ij}(t) = \int\int \mathcal{A}_\nu(l,m) I_\nu(l,m)\, e^{-2\pi i[u_{ij}(t)l + v_{ij}(t)m]} \mathrm{d}l\mathrm{d}m.$$

The observed visibilities, $\tilde{V}_{ij}(t)$, can be related to the true visibilities, V_{ij} through

$$\tilde{V}_{ij}(t) = G_{ij}(t)V_{ij}(t) + \varepsilon_{ij}(t) + n_{ij}(t),$$

where

$$G_{ij} = \text{baseline-based complex gain}$$
$$\varepsilon_{ij} = \text{baseline-based complex offset}$$
$$n_{ij} = \text{stochastic complex noise}$$

The complex offset, ε_{ij}, and complex noise, n_{ij}, are merely the complex resultants of the offsets and noises of two independent correlators and should not lower the coherence perceptibly.

Gain Calibration II

The baseline-based complex gain, G_{ij} can often be approximated by the product of the two associated **antenna-based complex gains**, g_i and g_j,

$$\boxed{G_{ij}(t) = g_i(t)g_j^*(t) = a_i(t)a_j(t)\, e^{i[\phi_i(t) - \phi_j(t)]}},$$

where $a_i(t)$ is the **gain amplitude** correction and $\phi_i(t)$ the **gain phase** correction.

In practice, the antenna-based gains are not only a function of time but also frequency

$$g_i(\nu, t) = g_i(t) g_i(\nu),$$

where $g_i(\nu)$ is the so-called passband gains, which describes the (assumed non-varying) response of the system across frequency bands. The process of solving for $g_i(\nu)$ is often referred to as the **bandpass calibration**.

From Visibility to Images

Observed quantities:

Visibilities $$V(u,v)S(u,v) = \int\int \mathcal{A}(l,m)I(l,m)\, e^{-2\pi i(ul+vm)} \mathrm{d}l\, \mathrm{d}m$$

Sampling function $$S(u,v) = \sum_{k=1}^{N} \delta(u-u_k, v-v_k)$$

Generating images from visibilities:

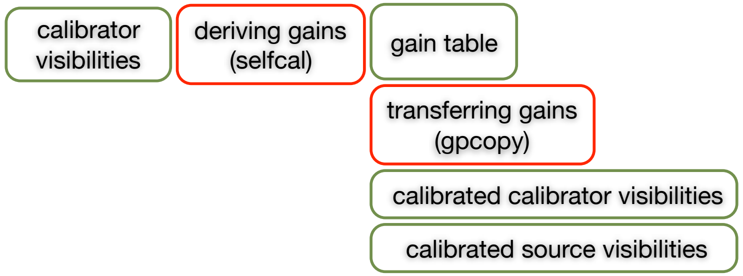
Synthesized beam $$B(l,m) = \int\int S(u,v)\, e^{2\pi i(ul+vm)} \mathrm{d}u\, \mathrm{d}v$$

Dirty map $$I^D(l,m) = [\mathcal{A}(l,m)I(l,m)] \otimes B(l,m) = \int\int V(u,v)S(u,v)\, e^{2\pi i(ul+vm)} \mathrm{d}u\, \mathrm{d}v$$

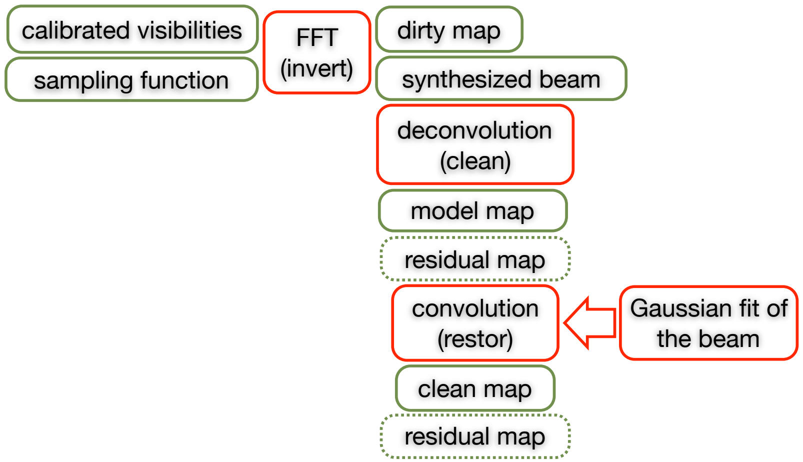
Weighted visibilities $$V^W(u,v) = \sum_{k=1}^{N} R_k T_k D_k V(u,v),$$

where R_k, T_k, and D_k are weights assigned to the visibilities indicating their reliability, the tapering, and the density weighting.

Visibility Calibration

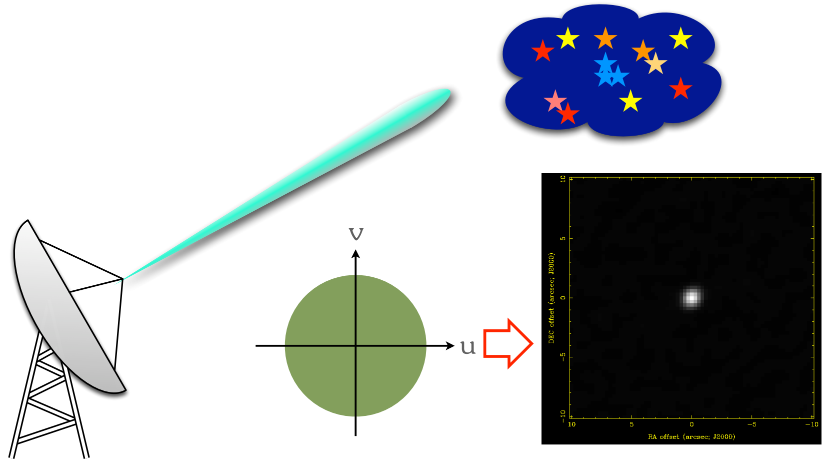


"Clean" Images



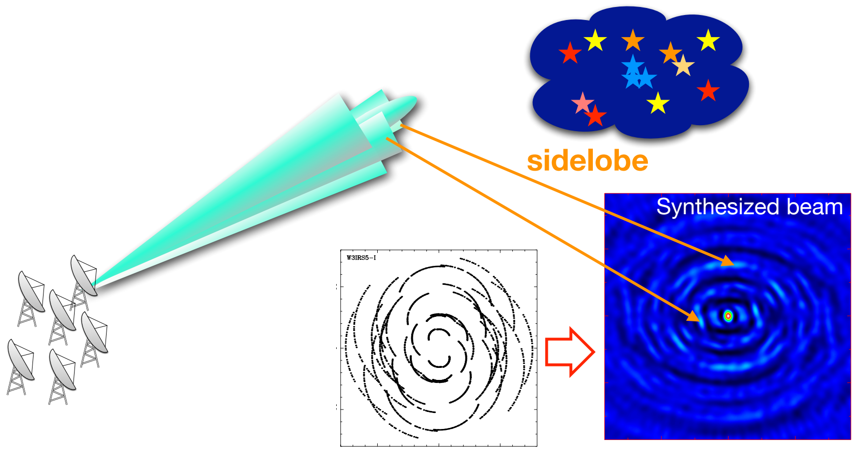
Why Restoring a Beam?

- **Single-dish comes with a more or less ideal beam**



Why Restoring a Beam?

- **Incomplete sampling with an array leads to a non-ideal beam with sidelobes**



Replacing Synthesized Beam

CLEAN Task in CASA

dirty map = sky brightness ⊗ synthesized beam

deconvolution means dirty map, synthesized beam ⇨ model map ≃ sky brightness

residual map ≃ error map

residual map = dirty map − model map ⊗ beam

convolution means

synthesized beam ⇨ Gaussian beam

clean map = model map ⊗ Gaussian beam + residual map

Dirty Map vs Clean Map