Introduction to General Relativity

Contexts

- Basic Property of Gravity
- Curved Spacetime
- Geodesic Deviation Equation and Curvature tensor
- Einstein equations
- Geometrical Interpretation of Ricc tensor and Weyl Tensor

1. From Special Relativity to General Relativity

We start from Minkowski spacetime .

Using an inerital coordinate $(X^{\bar{\alpha}}) = \left(X^{\bar{0}}, X^{\bar{\imath}}\right)$ the line element of the Minkowski spacetime may be written as

$$ds^2 = \eta_{\mu\nu} \; dX^{\bar{\alpha}} dX^{\bar{\beta}} = -(dX^{\bar{0}})^2 + (dX^{\bar{1}})^2 + (dX^{\bar{2}})^2 + (dX^{\bar{3}})^2$$

An inertial coordinate is the coordinate used by an inertia observer. , i.e
The observer without acceleration

η is called Minkowski metric tensor

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = diag(-1,+1,+1,+1)$$

Different inertial coordinates are related with each other by Lorentz transformation Λ

$$\eta_{\mu\nu} = \eta_{\alpha\beta} \; \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu}$$

Equivalence principle

Equation of Motion

$$m_I \vec{a} = m_G \vec{g}$$

m_I : Inertial mass

m_G : Gravitational mass(Gravitational charge)

Equivalence principle means $m_I = m_G$ ⟹ $\vec{a} = \vec{g}$

Gravity = Acceleration

One can expect that gravity may be expressed by the observer with acceleration(curved coordinate system) t

Line element in an accelerated frame

Instead of an inertia coordinate one can freely use any accelerated observer to set up the coordinate $(x^\mu) = (x^0, x^i)$

Since two events are covered by two coordinates $(X^{\bar{\alpha}})$ and (x^μ), we have

$$dX^{\bar{\alpha}}(x) = \frac{\partial X^{\bar{\alpha}}}{\partial x^\mu} dx^\mu \equiv e^{\bar{\alpha}}{}_\mu(x) dx^\mu$$

Thus we have the expression for the line element measured by an accelerated observer

$$ds^2 = \eta_{\mu\nu} dX^{\bar{\mu}} dX^{\bar{\nu}} = \eta_{\mu\nu}\; e^{\bar{\mu}}{}_\alpha \; e^{\bar{\nu}}{}_\beta \; dx^\alpha dx^\beta \equiv g_{\alpha\beta}(x) dx^\alpha dx^\beta$$

Metric tensor in the accelerated frame

$$g_{\alpha\beta}(x) = \eta_{\mu\nu}\; e^{\bar{\mu}}{}_\alpha(x)\; e^{\bar{\nu}}{}_\beta(x)$$

Example: Observer with Constant Acceleration

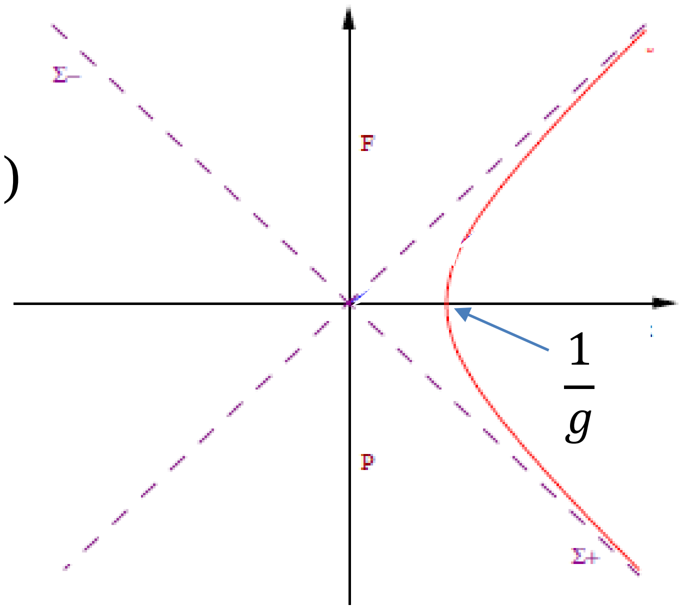
Consider the following worldline in Minkowski spacetime

$$X^{\bar{0}}(\tau) = g^{-1} \sinh(g\tau), \quad X^{\bar{1}}(\tau) = g^{-1} \cosh(g\tau), \quad -\infty < \tau < \infty$$

(g=const.)

4-velocity $\vec{U}$

$$U^{\bar{0}} = \frac{dX^{\bar{0}}}{d\tau} = \cosh(g\tau), \quad U^{\bar{1}} = \frac{dX^{\bar{1}}}{d\tau} = \sinh(g\tau)$$



3-velocity V

$$V = \frac{dX^{\bar{1}}}{dX^{\bar{0}}} = \frac{dX^{\bar{1}}/d\tau}{dX^{\bar{0}}/d\tau} = \frac{U^{\bar{1}}}{U^{\bar{0}}} = \tanh(g\tau) < 1$$

4-acceleration $\vec{a}$

$$a^{\bar{0}} = \frac{dU^{\bar{0}}}{d\tau} = g\sinh(g\tau), \quad a^{\bar{1}} = \frac{dU^{\bar{1}}}{d\tau} = g\cosh(g\tau)$$

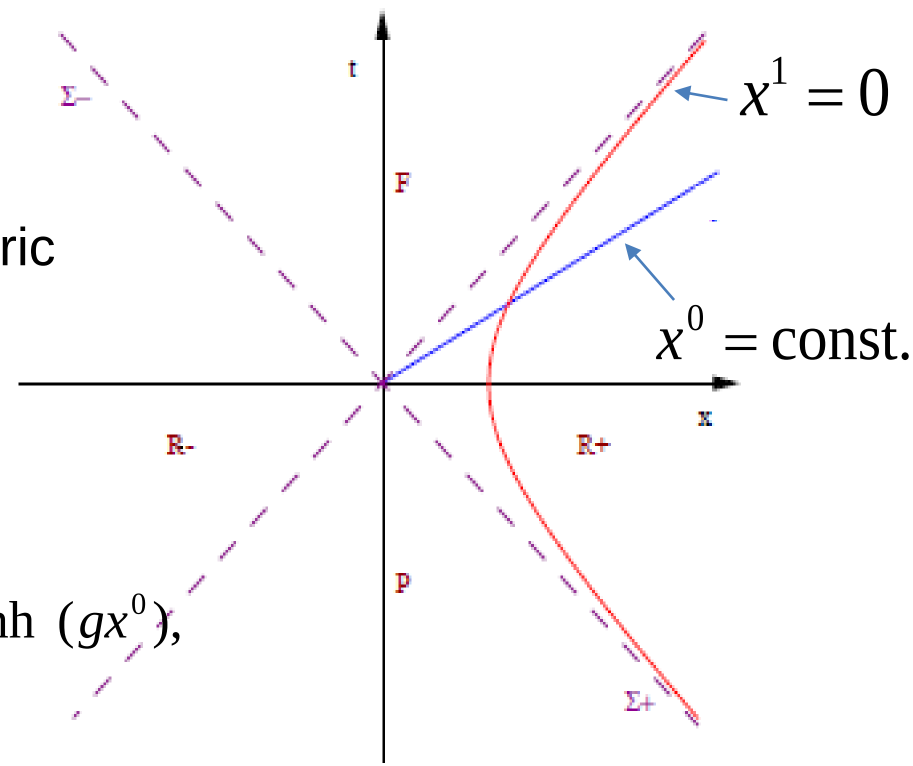
$$a^2 = \vec{a}\cdot\vec{a} = -(a^t)^2 + (a^x)^2 = g^2$$ Constant acceleration!

Consider the following coordinate system (x^0, x^1)

$$X^{\bar{0}}(x^0, x^1) = \left(x^1 + \frac{1}{g}\right)\sinh\,(gx^0), \quad X^{\bar{1}}(x^0, x^1) = \left(x^1 + \frac{1}{g}\right)\cosh\,(gx^0),$$

$$x^0 = \tau = \tanh \frac{X^{\bar{1}}}{X^{\bar{0}}},$$

$$x^1 + \frac{1}{g} = [(X^{\bar{1}})^2 - (X^{\bar{0}})^2]$$



Now we write the Minkowski metric in terms of this coordinate

$$dX^{\bar{0}} = \frac{\partial X^{\bar{0}}}{\partial x^0} dx^0 + \frac{\partial X^{\bar{0}}}{\partial x^1} dx^1$$

$$= \left(x^1 + \frac{1}{g}\right)\cosh\,(gx^0) + \sinh\,(gx^0),$$

$$dX^{\bar{1}} = \frac{\partial X^{\bar{1}}}{\partial x^0} dx^0 + \frac{\partial X^{\bar{1}}}{\partial x^1} dx^1$$

$$= \left(x^1 + \frac{1}{g}\right)\sinh\,(gx^0) + \cosh\,(gx^0)$$

$$e^{\hat{0}}{}_{\mu} = \frac{\partial X^{\hat{0}}}{\partial x^{\mu}} = \left(\left(x^1 + \frac{1}{g} \right) \cosh(gx^0), \sinh(gx^0) \right)$$

$$e^{\hat{1}}{}_{\mu} = \frac{\partial X^{\hat{1}}}{\partial x^{\mu}} = \left(\left(x^1 + \frac{1}{g} \right) \sinh(gx^0), \cosh(gx^0) \right)$$

New metric tensor takes the following expression

$$g_{00} = \eta_{\alpha\beta} e^{\bar{\alpha}}{}_0 e^{\bar{\beta}}{}_0 = -(e^{\bar{0}}{}_0)^2 + (e^{\bar{1}}{}_0)^2 = -\left(1 + gx^0\right)^2$$

$$g_{11} = \eta_{\alpha\beta} e^{\bar{\alpha}}{}_1 e^{\bar{\beta}}{}_1 = -(e^{\bar{0}}{}_1)^2 + (e^{\bar{1}}{}_1)^2 = 1$$

$$g_{01} = \eta_{\alpha\beta} e^{\bar{\alpha}}{}_0 e^{\bar{\beta}}{}_1 = -e^{\bar{0}}{}_0 e^{\bar{0}}{}_1 + e^{\bar{1}}{}_0 e^{\bar{1}}{}_1 = 0$$

Thus in the uniform accelerated coordinate the line element is written as

$$ds^2 = \eta_{\alpha\beta} e^{\bar{\alpha}}{}_{\mu} e^{\bar{\beta}}{}_{\nu} \; dx^{\mu} dx^{\nu} = -\left(1 + gx^1\right)^2 dx_0^2 + dx_1^2$$

This metric is equivalent to the metric of the neighborhood of Schwarzschild Black hole

In Schwarzschild coordinate, the Schwarzschild solution is written as

$$ds^2 = -\left(1-\frac{2GM}{r}\right)\ dt^2 + \frac{dr^2}{1-\frac{2GM}{r}} + r^2(d\theta^2 + \sin^2\theta\ d\varphi^2)$$

To see the neighborhood of the event horizon, we introduce the following new radial coordinate x

$$r = 2GM + x^2, \quad x^2 << 1$$

$$ds^2 = -\frac{x^2}{2GM + x^2} dt^2 + 4\frac{2GM + x^2}{x^2} x^2 dx^2 + r^2(d\theta^2 + \sin^2\theta\ d\varphi^2)$$

$$\approx -\frac{x^2}{2GM} dt^2 + 8GM\ dx^2 + (2GM + x^2)(d\theta^2 + \sin^2\theta\ d\varphi^2)$$

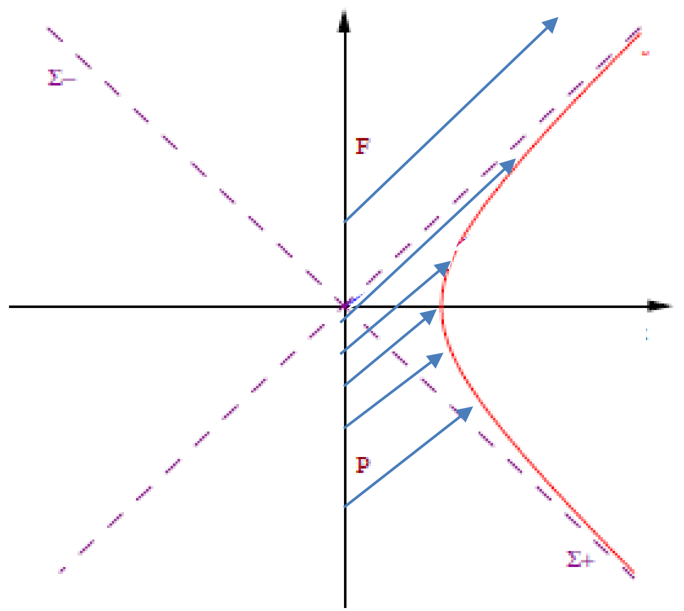
Compare this with the line element for the uniformly accelerated observaer

$$ds^2 = -\left(1 + gx^1\right)^2 dx_0^2 + dx_1^2$$

Doppler effect and event horizon

Energy of a photon k measured by an observer U

$$E = -\vec{k} \cdot \vec{U}$$



Our accelerated observer

$$U^{\mu}_{obs}(\tau) = (\ \cosh(g\tau),\ \sinh(g\tau),\ 0,\ 0\ )$$

Photons are continuously emitted to the observer from the origin $X^{\bar{1}} = 0$

$$\vec{k} = (\omega,\ \omega,\ 0,\ 0)$$

Observed energy

$$\omega_{obs} = -\vec{k} \cdot \vec{U}_{obs} = \ \omega\ \cosh(g\tau) - \omega\ \sinh(g\tau)$$

$$= \ \omega\ e^{-g\tau} \rightarrow 0 \quad as\ \tau \rightarrow \infty$$

This observer can never see any photon emitted after $X^{\bar{0}} > 0$.

Thus this observer can see only spacetime region $X^{\bar{1}} > X^{\bar{0}}$ even from infinite future. The boundary $X^{\bar{1}} = X^{\bar{0}}$ is called event horizon.

Equation of Motion(Geodesic equation)

Law of Inertia

A Particle with No External Force(Free Particle) moves on a straight line with a constant speed(Uniform Linear Motion).in any Inertial frame

$$\frac{d\vec{U}}{d\lambda} = 0 \qquad \text{in any inertial frame}$$

$\vec{U} = \frac{d\vec{x}}{d\lambda}$: 4 velocity along a world line $\vec{x} = \vec{x}(\lambda)$

$\vec{x} = \vec{x}(\lambda)$

$$X^{\bar{\alpha}} = X^{\bar{\alpha}}(\lambda), \; U^{\bar{\alpha}} = \frac{dX^{\bar{\alpha}}}{d\lambda} \qquad \text{Inertial frame}$$

$$x^{\mu} = x^{\mu}(\lambda), \quad U^{\mu} = \frac{dx^{\mu}}{d\lambda} \qquad \text{Accelerated frame}$$

Relation of components between inertial and accelerated frame

$$U^{\bar{\alpha}} = \frac{dX^{\bar{\alpha}}}{d\lambda} = \frac{\partial X^{\bar{\alpha}}}{\partial x^{\mu}} \frac{dx^{\mu}}{d\lambda} = e^{\bar{\alpha}}{}_{\mu}(x) U^{\mu}$$

4-vector in an accelerated frame

$$U^{\mu} = e^{\mu}_{\bar{\alpha}} U^{\bar{\alpha}} \quad \text{with } e^{\mu}_{\bar{\alpha}} = \frac{\partial x^{\mu}}{\partial X^{\bar{\alpha}}}$$

Thus the law of Inertia takes the following form in an accelerated frame

$$0 = \frac{dU^{\hat{\alpha}}}{d\lambda} = \frac{d}{d\lambda}\left(e^{\hat{\alpha}}{}_{\mu}\, U^{\mu}\right) = e^{\hat{\alpha}}{}_{\mu}\ \frac{dU^{\mu}}{d\lambda} + \frac{de^{\hat{\alpha}}{}_{\mu}}{d\lambda} U^{\mu}$$

Since d/dλ is the derivative along the world line, we can write

$$\frac{de^{\bar{\alpha}}{}_{\mu}}{d\lambda} = \frac{dx^{\nu}}{d\tau}\frac{\partial e^{\bar{\alpha}}{}_{\mu}}{\partial x^{\nu}} = U^{\nu}\ \frac{\partial e^{\bar{\alpha}}{}_{\mu}}{\partial x^{\nu}} = U^{\nu}\Gamma^{\rho}{}_{\mu\nu} e^{\bar{\alpha}}{}_{\rho}$$

Here we have introduced the concept of connection Γ as follows

$$e^{\bar{\alpha}}{}_{\mu}(x+dx) = e^{\bar{\alpha}}{}_{\mu}(x) + \Gamma^{\rho}{}_{\mu\nu}(x) e^{\bar{\alpha}}{}_{\rho}(x) dx^{\nu} \Rightarrow \frac{\partial e^{\bar{\alpha}}{}_{\mu}}{\partial x^{\nu}} = \Gamma^{\rho}{}_{\mu\nu} e^{\bar{\alpha}}{}_{\rho}$$

Thus we have

$$e^{\hat{\alpha}}{}_{\mu}\ \frac{dU^{\mu}}{d\lambda} + U^{\nu}\Gamma^{\rho}{}_{\mu\nu} e^{\hat{\alpha}}{}_{\rho} U^{\mu} = 0$$

$$e^{\hat{\alpha}}{}_{\rho}\ \left(\frac{dU^{\rho}}{d\lambda} + U^{\nu}\Gamma^{\rho}{}_{\mu\nu} U^{\mu}\right) = 0$$

Multiplying both side by the inverse matrix $e^{\rho}{}_{\bar{\alpha}}=\frac{\partial x^{\rho}}{\partial x^{\bar{\alpha}}}$,

$$\frac{dU^{\mu}}{d\lambda}+\Gamma^{\mu}{}_{\rho\sigma}U^{\rho}U^{\sigma}=0$$

We can write this equation as

$$\text{l.h.s}=\frac{dx^{\rho}}{d\lambda}\frac{\partial U^{\mu}}{\partial x^{\rho}}+\Gamma^{\mu}{}_{\rho\sigma}U^{\rho}U^{\sigma}=U^{\rho}\left(\frac{\partial U^{\mu}}{\partial x^{\rho}}+\Gamma^{\mu}{}_{\rho\sigma}U^{\sigma}\right)\equiv U^{\rho}\nabla_{\rho}U^{\mu}$$

Thus the law of inertia takes the following form in an accelerated frame

$$U^{\rho}\nabla_{\rho}U^{\mu}=0$$

Compare this with the equation in an inertial frame

$$U^{\bar{\alpha}}\partial_{\bar{\alpha}}U^{\bar{\beta}}=0$$

Thus the law takes the same form if we use the covariant derivative instead of partial derivative

$$\partial_{\bar{\alpha}}\rightarrow\nabla_{\mu}$$

This is mathematical expression of Einstein’s relativity principle

Properties of Γ

1. Symmetry

$$e^{\bar{\alpha}}{}_{\mu}(x) \equiv \frac{\partial X^{\bar{\alpha}}}{\partial x^{\mu}}, \quad \Rightarrow \quad \Gamma^{\rho}{}_{\mu\nu} = e^{\rho}{}_{\bar{\alpha}} \frac{\partial^2 X^{\bar{\alpha}}}{\partial x^{\mu} \partial x^{\nu}} = \Gamma^{\rho}{}_{\nu\mu} \quad \text{where} \quad e^{\rho}{}_{\bar{\alpha}} = \frac{\partial x^{\rho}}{\partial X^{\bar{\alpha}}}$$

2. The expression by the metric tensor

$$g_{\mu\nu} = \eta_{\alpha\beta} e^{\bar{\alpha}}{}_{\mu}\, e^{\bar{\beta}}{}_{\nu}$$

$$\frac{\partial g_{\mu\nu}}{\partial x^{\rho}} = \left(e^{\hat{\alpha}}{}_{\mu,\rho}\, e^{\hat{\beta}}{}_{\nu} + e^{\hat{\alpha}}{}_{\mu}\, e^{\hat{\beta}}{}_{\nu,\rho} \right) \eta_{\alpha\beta}$$

$$= \left(\Gamma^{\sigma}{}_{\mu\rho}\, e^{\hat{\alpha}}{}_{\sigma}\, e^{\hat{\beta}}{}_{\nu} + e^{\hat{\alpha}}{}_{\mu}\, \Gamma^{\sigma}{}_{\nu\rho}\, e^{\hat{\beta}}{}_{\sigma} \right) \eta_{\alpha\beta}$$

$$= \Gamma^{\sigma}{}_{\mu\rho}\, g_{\sigma\nu} + \Gamma^{\sigma}{}_{\nu\rho}\, g_{\mu\sigma}$$

$$\Rightarrow \quad \frac{\partial g_{\mu\nu}}{\partial x^{\rho}} + \frac{\partial g_{\rho\mu}}{\partial x^{\nu}} - \frac{\partial g_{\nu\rho}}{\partial x^{\mu}} = 2 g_{\mu\sigma} \Gamma^{\sigma}{}_{\rho\nu}$$

$$\Rightarrow \quad \Gamma^{\sigma}{}_{\rho\nu} = \frac{1}{2} g^{\sigma\lambda} \left(\frac{\partial g_{\lambda\nu}}{\partial x^{\rho}} + \frac{\partial g_{\lambda\mu}}{\partial x^{\nu}} - \frac{\partial g_{\nu\rho}}{\partial x^{\lambda}} \right)$$

Let's calculate equation of motion observed by an accelerated observer

$$ds^2 = -\left(1+gx^1\right)^2 dx_0^2 + dx_1^2$$

Christoffel symbol are

$$\Gamma^0{}_{\alpha\beta} = \frac{1}{2} g^{00}\left(\frac{\partial g_{0\beta}}{\partial x^\alpha} + \frac{\partial g_{\alpha 0}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^0}\right) \Rightarrow \Gamma^0{}_{01} = \frac{1}{2} g^{00} \frac{\partial g_{00}}{\partial x^1} = (1+gx^1)^{-1} g$$

$$\Gamma^1{}_{\alpha\beta} = \frac{1}{2} g^{11}\left(\frac{\partial g_{1\beta}}{\partial x^\alpha} + \frac{\partial g_{\alpha 1}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^1}\right) \Rightarrow \Gamma^1{}_{00} = -\frac{1}{2} g^{11} \frac{\partial g_{00}}{\partial x^1} = (1+gx^1) g$$

We use proper time τ as the parameter

$$d\tau^2 = -ds^2 = \left(1+gx^1\right)^2 \left(dx_0\right)^2 - \left(dx_1\right)^2 = \left(dx_0\right)^2 \left[\left(1+gx^1\right)^2 - V^2\right], \quad V \equiv \frac{dx^1}{dx^0}$$

$$\Rightarrow \tau = x^0 \sqrt{\left(1+gx^1\right)^2 - V^2} \Rightarrow x^0 = \frac{\tau}{\sqrt{\left(1+gx^1\right)^2 - V^2}} \approx \tau\left(1 - gx^1 + \frac{1}{2}V^2\right)$$

Then the 4-velocity becomes

$$U^\mu = \frac{dx^\mu}{d\tau} = \frac{1}{\sqrt{(1+gx^1)^2 - V^2}}\left(1, \ V\right)$$

Now consider Newtonian situation where g<<1, V^2<<1

$$ds^2 = -\left(1 + 2gx^1\right)dx_0^2 + dx_1^2$$

$$\Gamma^0{}_{01} \approx g, \quad \Gamma^1{}_{00} \approx g$$

$$U^\mu \approx \left(1,\ V\right)$$

0th component of geodesic equation

$$\frac{dU^0}{d\tau} + 2\Gamma^0{}_{01}U^0U^1 = 0 \Rightarrow 0 = 0$$

1st component of geodesic equation

$$\frac{dU^1}{d\tau} + \Gamma^1{}_{00}U^0U^0 = 0 \Rightarrow \frac{dV}{dt} + g = 0$$

Newton’s EOM

Newtonian Limit in general

Spacetime is nearly flat and the typical velocity of the system is much smaller than the speed of light

$$g = \eta + h, \quad |h| << 1$$

Proper time

$$d\tau^2 = -g_{\alpha\beta} dx^\alpha dx^\beta = -(\eta_{\alpha\beta} + h_{\alpha\beta}) dx^\alpha dx^\beta$$

$$= dt^2 - d\vec{x}^2 - h_{\alpha\beta} dx^\alpha dx^\beta$$

$$= dt^2 \left[1 - v^2 - h_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} \right] \Rightarrow d\tau = dt + O(v^2, \ h)$$

Thus the 4-velocity takes the following form.

$$U^\alpha = \frac{dx^\alpha}{d\tau} \approx \frac{dx^\alpha}{dt} = (1, \ v^i)$$

$$\frac{dU^\alpha}{d\tau} + \Gamma^\alpha{}_{\rho\sigma} U^\rho U^\sigma = 0 \quad \Rightarrow \quad \frac{dv^i}{d\tau} + \Gamma^i{}_{00} = 0$$

$$\Gamma^i{}_{00} = \frac{1}{2} g^{i\mu} (g_{\mu 0,0} + g_{0\mu,0} - g_{00,\mu})$$

$$= \frac{1}{2} g^{ik} (2 g_{k0,0} - g_{00,k}) \approx -\frac{1}{2} \delta^{ik} h_{00,k}$$

$$\Rightarrow \quad \frac{dv^i}{d\tau} = \frac{1}{2} \delta^{ik} h_{00,k}$$

Where we neglect the term with time derivative compared with the term with spatial derivative

$$\frac{A_{,0}}{A_{,k}} \approx \frac{A/T}{A/L} \approx \frac{L}{T} \approx v << 1$$

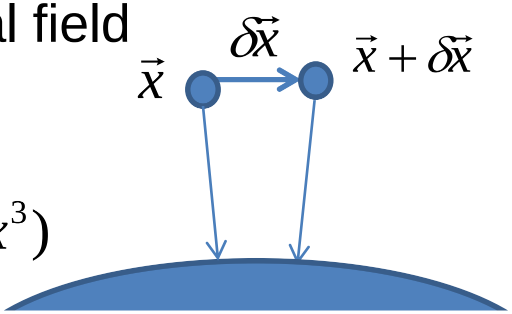
Thus we recover the Newtonian equation if we take h_{00} as -2ϕ (ϕ: Newtonian gravitational potential)

$$\boxed{g_{00} \approx -1 + 2\phi}$$

$$\frac{d}{d\tau} v^i = -\phi^{,i}$$

Flat Minkowski spacetime is not enough to describe gravity

Acceleration can produce only uniform gravitational field



$$\ddot{x}^i = \partial_i \phi(\vec{x})$$

$$\ddot{x}^i + \delta\,\ddot{x}^i = \partial_i \phi(\vec{x}+\delta\vec{x}) = \partial_i \phi(\vec{x}) + \partial_{ij}\phi(\vec{x})\delta x^j + O(\delta x^3)$$

$$\delta\,\ddot{x}^i \approx \partial_{ij}\phi(\vec{x})\delta x^j$$

Non-uniform gravity appears as the difference of inertial frame at different spacetime point(event)

Einstein regards this difference as the effect of curvature of spacetime

Thus we need curved spacetime instead of Minkowski spacetime when gravitational field exist

How to describe Curved Spacetime

Line element of spacetime

$$ds^2 = g_{\mu\nu}(x)\; dx^\mu dx^\nu$$

Global inertial frames exit In Special Relativity because the spacetimes is globally flat

In curved spacetime inertial frames exit only locally (Local inertial frame)

Local inertial frame around an event p is characterized by the following properties

$$g_{\bar{\alpha}\bar{\beta}}(p) = \eta_{\alpha\beta}$$

$$g_{\bar{\alpha}\bar{\beta},\bar{\rho}}(p) = 0$$

$$g_{\bar{\alpha}\bar{\beta},\bar{\rho}\bar{\sigma}}(p) \neq 0 \;\Longrightarrow\; \text{Effect of Curvature}$$

Knowledge of the metric tensor is not enough to say if the spacetime is flat or curved

Simple example: 2D sphere S^2

In spherical coordinate (θ, φ) the distance between neighboring points with coordinate difference $(d\theta, d\varphi)$ t on sphere of radius a is

$$ds^2 = a^2 (d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Introduce new coordinate around a point O $\left(\theta = \frac{\pi}{2}, \varphi = 0\right)$

$$x^{1'} = a(\theta - \pi/2), \qquad x^{2'} = a\varphi$$

In this coordinate the distance is written as

$$ds^2 = (dx^{1'})^2 + \cos^2\left(\frac{x^{1'}}{a}\right)(dx^{2'})^2$$

$$g'_{11} = 1, \;\; g'_{12} = g'_{21} = 0, \;\; g'_{22} = \cos^2\left(\frac{x^{1'}}{a}\right)$$

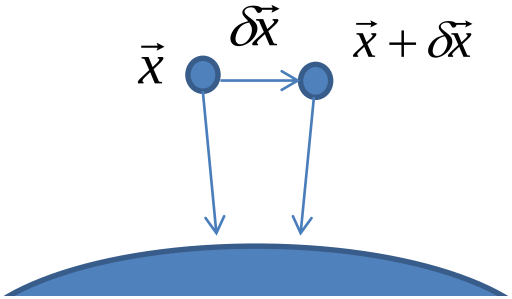
In the neighborhood of point O: $|\, x^{i'} / a \,| << 1$

$$g'_{22} = \cos^2\left(\frac{x^{1'}}{a}\right) \approx 1 - \left(\frac{x^{1'}}{a}\right)^2 + O\left(\left(\frac{x^{1'}}{a}\right)^4\right) \Rightarrow g_{22,11}(O) = -\frac{2}{a^2}$$

In Newtonian case the difference of two neighboring two inertial frames are described by tidal force

$$A: \frac{d^2}{dt^2} x^i = -\frac{\partial \phi(x)}{\partial x^i}$$

$$B: \frac{d^2}{dt^2} (x^i + \zeta^i) = -\frac{\partial \phi(x+\zeta)}{\partial x^i}$$

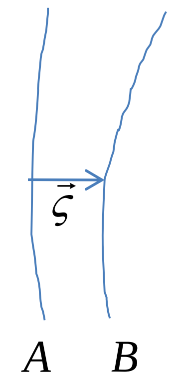


$$\Rightarrow \quad B-A: \frac{d^2}{dt^2} \zeta^i = -\frac{\partial^2 \phi(x)}{\partial x^i \partial x^k} \zeta^k \quad \text{Tidal force}$$

Corresponding calculation in curved spaceime

$$A: x^\mu(\lambda) \quad \text{geodesic through } A$$

$$B: x^\mu(\lambda) + \zeta^\mu(\lambda) \quad \text{geodesic through } B$$



$$\Rightarrow \quad B-A: \frac{d^2}{d\lambda^2} \zeta^\mu + \zeta^\rho \Gamma^\mu{}_{\alpha\beta,\rho} \frac{dx^\alpha}{d\lambda} \frac{dx^\alpha}{d\lambda} + 2\Gamma^\mu{}_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{d\zeta^\alpha}{d\lambda} = 0$$

We have to make this equation into the covariant form (the form looks same in any coordinate system)

$$\frac{d}{d\lambda} = u^\alpha \partial_\alpha \to \frac{D}{D\lambda} = u^\alpha \nabla_\alpha$$

The covariant expression

$$\frac{D^2}{D\lambda^2}\zeta^\mu = (\underbrace{\Gamma^\mu{}_{\beta\nu,\alpha} - \Gamma^\mu{}_{\alpha\nu,\beta} + \Gamma^\mu{}_{\alpha\delta}\Gamma^\delta{}_{\beta\nu} - \Gamma^\mu{}_{\beta\delta}\Gamma^\delta{}_{\alpha\nu}}_{R^\mu{}_{\nu\alpha\beta}})\, u^\alpha u^\nu \zeta^\beta$$

$R^\mu{}_{\nu\alpha\beta}$ Riemann tensor

$$R^\mu{}_{\alpha\nu\beta} = \Gamma^\mu{}_{\beta\nu,\alpha} - \Gamma^\mu{}_{\alpha\nu,\beta} + \Gamma^\mu{}_{\alpha\delta}\Gamma^\delta{}_{\beta\nu} - \Gamma^\mu{}_{\beta\delta}\Gamma^\delta{}_{\alpha\nu}$$

Also the definition of the covariant derivative for vector gives the following formula

$$\nabla_{[\alpha}\nabla_{\beta]}\, u^\mu \equiv (\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)\, u^\mu = R^\mu{}_{\nu\alpha\beta}\, u^\nu$$

Ricci scalar

$$R_{\nu\beta} \equiv R^\mu{}_{\nu\mu\beta} = g^{\mu\alpha} R_{\mu\nu\alpha\beta}$$

Ricci Tensor

$$R \equiv R^\alpha{}_\alpha = g^{\alpha\beta} R_{\alpha\beta}$$

Detailed calculation

$$\frac{D^2}{D\lambda^2}\zeta^\mu = u^\alpha \nabla_\alpha (u^\beta \nabla_\beta \zeta^\mu)$$

$$= u^\alpha \partial_\alpha (u^\beta \nabla_\beta \zeta^\mu) + u^\alpha \Gamma^\mu{}_{\alpha\gamma} u^\beta \nabla_\beta \zeta^\mu$$

$$= \frac{d}{d\lambda}\left(\frac{d}{d\lambda}\zeta^\mu + \Gamma^\mu{}_{\beta\gamma}\zeta^\beta u^\gamma \right) + u^\alpha \Gamma^\mu{}_{\alpha\gamma}\left(\frac{d}{d\lambda}\zeta^\gamma + \Gamma^\gamma{}_{\beta\delta}\zeta^\beta u^\delta \right)$$

$$= \frac{d^2\zeta^\mu}{d\lambda^2} + u^\alpha \Gamma^\mu{}_{\beta\gamma,\alpha}\zeta^\beta u^\gamma + \Gamma^\mu{}_{\beta\gamma}\frac{d\zeta^\beta}{d\lambda}u^\gamma + \Gamma^\mu{}_{\beta\gamma}\zeta^\beta \frac{du^\gamma}{d\lambda}$$

$$+ u^\alpha \Gamma^\mu{}_{\alpha\gamma}\frac{d\zeta^\gamma}{d\lambda} + u^\alpha \Gamma^\mu{}_{\alpha\gamma}\Gamma^\gamma{}_{\beta\delta}\zeta^\beta u^\delta$$

$$= \frac{d^2\zeta^\mu}{d\lambda^2} + u^\alpha \Gamma^\mu{}_{\beta\gamma,\alpha}\zeta^\beta u^\gamma + 2u^\alpha \Gamma^\mu{}_{\alpha\gamma}\frac{d\zeta^\gamma}{d\lambda} - \Gamma^\mu{}_{\beta\gamma}\zeta^\beta \Gamma^\gamma{}_{\alpha\delta}u^\alpha u^\delta + u^\alpha \Gamma^\mu{}_{\alpha\gamma}\Gamma^\gamma{}_{\beta\delta}\zeta^\beta u^\delta$$

$$= -\zeta^\gamma \Gamma^\mu{}_{\alpha\beta,\gamma}u^\alpha u^\beta - 2\Gamma^\mu{}_{\alpha\beta}u^\alpha \frac{d\zeta^\beta}{d\lambda}$$

$$+ u^\alpha \Gamma^\mu{}_{\gamma\beta,\alpha}\zeta^\gamma u^\beta + 2u^\alpha \Gamma^\mu{}_{\alpha\gamma}\frac{d\zeta^\gamma}{d\lambda} - \Gamma^\mu{}_{\beta\gamma}\zeta^\beta \Gamma^\gamma{}_{\alpha\delta}u^\alpha u^\delta + u^\alpha \Gamma^\mu{}_{\alpha\gamma}\Gamma^\gamma{}_{\beta\delta}\zeta^\beta u^\delta$$

Newtonian limit

$$\frac{D^2}{D\lambda^2}\zeta^\mu = R^\mu{}_{\nu\alpha\beta}u^\alpha u^\nu \zeta^\beta$$

$$\lambda \to t, \quad u^\alpha \approx (1,0,0,0)$$

$$\frac{\partial^2}{\partial t^2}\zeta^i \approx R^i{}_{00j}\zeta^j$$

$$g_{00} \approx -1 - 2\phi$$

$$R^i{}_{00j} \to \frac{\partial \phi}{\partial x^i \partial x^j}$$

Symmetry of Riemann Tensor

1. $R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu}$
2. $R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$
3. $R_{\alpha\beta\mu\nu} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\beta\mu} = 0 \Leftrightarrow R_{\alpha[\beta\mu\nu]} = 0$

Only 2 in 3 conditions are independent

For example the condition2. is the result from the condition 1 and 3

$$2R_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - R_{\beta\alpha\mu\nu}$$

$$= R_{\alpha\beta\mu\nu} + R_{\beta\mu\nu\alpha} + R_{\beta\nu\alpha\mu}$$

$$= R_{\alpha\beta\mu\nu} - R_{\mu\beta\nu\alpha} - R_{\nu\beta\alpha\mu}$$

$$= R_{\alpha\beta\mu\nu} + (R_{\mu\nu\alpha\beta} + R_{\mu\alpha\beta\nu}) + (R_{\nu\alpha\mu\beta} + R_{\nu\mu\beta\alpha})$$

$$= 2R_{\mu\nu\alpha\beta} + \underbrace{(R_{\alpha\beta\mu\nu} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\beta\mu})}_{0}$$

Number of Independent components of Riemann Tensor

$R_{\alpha\beta\mu\nu} = R_{[\alpha\beta][\mu\nu]} \rightarrow$ Number of pair $[\alpha,\beta]$ is $\frac{1}{2}$n(n-1)

$R_{\alpha[\beta\mu\nu]} = 0 \quad \rightarrow$ Number of the condition is $n\ {}_nC_3$

In n-dimensional spacetime

$$\left(\frac{n(n-1)}{2}\right)^2 - n\frac{n(n-1)(n-2)}{3!} = \frac{1}{12}n^2(n^2-1)$$

Thus indep. number of Riemann tensor in 4-dim. Is 20

Ricci Tensor

$R_{\mu\nu} = R_{\mu\nu} \Rightarrow$ Indep. number of Ricci tensor is $\frac{1}{2}$ n(n+1)

Thus indep. number of Ricci tensor in 4-dim. is 10

Spacetime is flat if and only if Riemann tensor vanishes.

Einstein equation in vacuum $R_{\mu\nu} = 0$

⇨ Vacuum spacetime is not necessarily flat! Black holes, GW

Weyl Tensor(Conformal Tensor) (n>3)

Riem=Ricci+Weyl

$$C_{\alpha\beta\mu\nu} \equiv R_{\alpha\beta\mu\nu} - \frac{1}{n-2}\left(g_{\alpha\mu}R_{\beta\nu} - g_{\alpha\nu}R_{\beta\mu} - g_{\beta\mu}R_{\alpha\nu} + g_{\beta\nu}R_{\alpha\mu}\right)$$
$$+\frac{R}{(n-1)(n-2)}\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right)$$

Weyl Tensor has same symmetry with Riemann Tensor and the following symmetry

$$C^{\alpha}{}_{\alpha\mu\nu} = C_{\alpha\beta}{}^{\mu}{}_{\mu} = 0$$

Independent component of Weyl Tenor

$$\frac{1}{12}n^2(n^2-1) - \frac{1}{2}n(n+1) = \frac{1}{12}n(n+1)(n+2)(n+3)$$

Black hole solution and Gravitational wave solution of Einstein equation have non-zero Weyl Tensor

Bianchi Identity

$$R_{\alpha\beta\mu\nu,\lambda} + R_{\alpha\beta\lambda\mu,\nu} + R_{\alpha\beta\nu\lambda,\mu} = 0$$

This identity can be easily confirmed in an inertial frame

$$R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}{}_{\beta\nu,\mu} - \Gamma^{\alpha}{}_{\beta\mu,\nu} = \frac{1}{2} g^{\alpha\sigma}(g_{\sigma\nu,\beta\mu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu} - g_{\beta\nu,\sigma\mu})$$

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(g_{\alpha\nu,\beta\mu} - g_{\alpha\mu,\beta\nu} + g_{\beta\mu,\alpha\nu} - g_{\beta\nu,\alpha\mu})$$

Contracted Bianchi identity

$$g^{\alpha\mu}(R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu}) = 0$$

$$\Rightarrow \quad R_{\beta\nu;\lambda} - R_{\beta\lambda;\nu} + R^{\alpha}{}_{\beta\nu\lambda;\alpha} = 0$$

$$g^{\beta\nu}(R_{\beta\nu;\lambda} - R_{\beta\lambda;\nu} + R^{\alpha}{}_{\beta\nu\lambda;\alpha}) = 0$$

$$R_{;\lambda} - R^{\beta}{}_{\lambda;\beta} - R^{\alpha}{}_{\lambda;\alpha} = 0 \Leftrightarrow G^{\alpha}{}_{\lambda;\alpha} = 0$$

Contracted Bianchi Identity

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0$$

$$\Rightarrow \quad g^{\alpha\mu}(R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu}) = 0$$

$$R_{\beta\nu;\lambda} - R_{\beta\lambda;\nu} + R^{\alpha}{}_{\beta\nu\lambda;\alpha} = 0$$

$$g^{\beta\nu}(R_{\beta\nu;\lambda} - R_{\beta\lambda;\nu} + R^{\alpha}{}_{\beta\nu\lambda;\alpha}) = 0$$

$$\Downarrow$$

$$R_{;\lambda} - R^{\beta}{}_{\lambda;\beta} - R^{\alpha}{}_{\lambda;\alpha} = 0 \Leftrightarrow G^{\alpha}{}_{\lambda;\alpha} = 0$$

Einstein Tensor : $\quad G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$

Geometrical Meaning of Riemann and Weyl

Propagation of light bundle in gravitational field

The propagation is described by Geodesic deviation equation

$$\frac{D^2}{D\lambda^2}\zeta^{\mu} = R^{\mu}{}_{\nu\alpha\beta}k^{\nu}k^{\alpha}\zeta^{\beta}, \qquad k^2 = 0, \qquad k^{\alpha}\nabla_{\alpha}k^{\mu} = 0,$$

Consider 2-dimensional spatial plane perpendicular to the direction of light propagation

Unit orthonormal base in this 2-plene

$$\vec{e}_a = (e^{\mu}{}_1,\ e^{\mu}{}_2) \perp \vec{k}, \qquad e^{\mu}{}_a k_{\mu} = 0,\ \ e^{\mu}{}_a e_{b\mu} = \delta_{ab}$$

$$\frac{D}{D\lambda}e^{\mu}{}_a = k^{\alpha}\nabla_{\alpha}e^{\mu}{}_a = 0$$

Transport this vector parallelly along $\vec{k}$

Then we project 4-vector $\vec{\zeta}$ into this plane

$$\zeta^{\mu}(\lambda) = \sum_{a=1}^{2}\ell_a(\lambda)\ e^{\mu}{}_a$$

Then we can derive the following

$$\frac{d^2\ell_a}{d\lambda^2} = -\frac{1}{2}R_{\mu\nu}k^\mu k^\nu \ell_a + \sum_b C_{\mu\nu\alpha\beta}\, e^\mu{}_a k^\nu k^\alpha e^\beta{}_b\, \ell_b$$

To see the change of the area more clearly we introduce the following

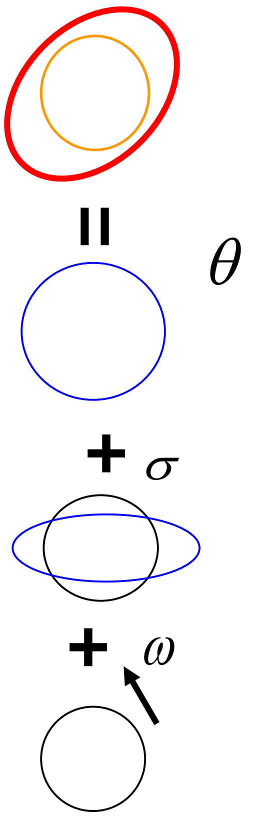
$$\frac{d\ell_a}{d\lambda} \equiv \sum_b (\ \theta\,\delta_{ab} + \sigma_{ab} + \omega_{ab})\ell_b$$

$$\sigma_{ab} = \sigma_{ba}, \qquad \mathrm{Tr}\,\sigma = \sigma_{11} + \sigma_{22} = 0, \qquad \omega_{ab} = -\omega_{ba}$$

Then the change of θ and σ along the light ray can be written as

$$\frac{d\theta}{d\lambda} + \theta^2 + \sigma^2 - \omega^2 = -\frac{1}{2}R_{\mu\nu}k^\mu k^\nu$$

$$\frac{d\sigma_{ab}}{d\lambda} + 2\theta\sigma_{ab} = C_{\mu\nu\alpha\beta}e^\mu{}_a k^\nu k^\alpha e^\beta{}_b$$



The area of the cross section of the light bundle changes according to the Ricci tensor, and the deformation of the cross section is generated by Weyl tensor

$$\frac{d\theta}{d\lambda} + \theta^2 = -\frac{1}{2} R_{\mu\nu} k^\mu k^\nu - \sigma^2$$

Ricci Forcussing Weyl Forcussing

Acutually Ricci forcussing is generated by the metter distribution through Einstein equation

$$R_{\mu\nu} = 8\pi\ G\ (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T)$$

$$\boxed{\frac{d\theta}{d\lambda} + \theta^2 = -4\pi G T_{\mu\nu} k^\mu k^\nu - \sigma^2}$$

This is the starting equation to derive the formula for distance fluctuation by inhomogeneous matter distribution

$$\delta_d(z_s, \mathbf{n}) \equiv \frac{\delta d_L^{FRW}(z_s, \mathbf{n})}{d_L^{FRW}(z_s)} = -\int_0^{\chi_s} d\chi \frac{(\chi_s - \chi)\chi}{\chi_s} \left(4\pi G a^2 \delta\rho_m + \sigma^2\right) +$$

Detailed calculation.

First we prove that vector $\vec{\zeta}$ can be taken to be perpendicular to the direction of propagation $\vec{k}$.

$$k^\alpha \nabla_\alpha (k \cdot \zeta) = (k^\alpha \nabla_\alpha k^\mu)\zeta_\mu + k^\alpha k^\mu \nabla_\alpha \zeta_\mu = 0$$

$$\because \quad g_{\mu\nu}(x+\zeta)\left(k^\mu + \frac{d\zeta^\mu}{d\lambda}\right)\left(k^\nu + \frac{d\zeta^\nu}{d\lambda}\right) = 0 \Rightarrow \zeta^\alpha g_{\mu\nu,\rho} k^\mu k^\nu + 2 g_{\mu\nu} k^\mu \frac{d\zeta^\nu}{d\lambda} = 0$$

$$\Leftrightarrow k^\mu k^\nu \nabla_\nu \zeta_\mu = 0$$

Substitute $\zeta^\mu(\lambda) = \sum_{a=1}^{2} \ell_a(\lambda)\ e^\mu{}_a$ into geodesic deviation equation.

$$\frac{D^2}{D\lambda^2}\varsigma^\mu = \sum_a \frac{d^2 \ell_b}{d\lambda^2}\ e^\mu{}_b = R^\mu{}_{\nu\alpha\beta} k^\nu k^\alpha \sum_b \ell_b\ e^\beta{}_b$$

$$e_{a\mu} \times \qquad \frac{d^2 \ell_a}{d\lambda^2} = R^\mu{}_{\nu\alpha\beta} k^\nu k^\alpha \sum_b \ell_b\ e^\beta{}_b\ e_{a\mu} \equiv \sum_b K_{ab} \ell_b$$

where $\quad K_{ab} \equiv R_{\mu\nu\alpha\beta} e^\mu{}_a k^\nu k^\alpha e^\beta{}_b = K_{ba}$

K_{ab} can be expressed as the sum of Ricci part and Weyl part as follows.

$$K_{ab} = -\frac{1}{2} R_{\nu\alpha} k^{\nu} k^{\alpha} \delta_{ab} + C_{\mu\nu\alpha\beta} e^{\mu}{}_{a} k^{\nu} k^{\alpha} e^{\beta}{}_{b}$$

Where we have used

$$R_{\mu\nu\alpha\beta} = \frac{1}{2}(g_{\mu\alpha} R_{\nu\beta} - g_{\mu\beta} R_{\nu\alpha} - g_{\nu\alpha} R_{\mu\beta} + g_{\nu\beta} R_{\mu\alpha})$$

$$-\frac{1}{6} R \,(g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}) + C_{\mu\nu\alpha\beta}$$

$$k^2 = 0, \quad k^{\mu} e_{a\mu} = 0$$

Thus the change of the 2-vectors ℓ_a along the light ray can be written as follows

$$\frac{d^2 \ell_a}{d\lambda^2} = -\frac{1}{2} R_{\mu\nu} k^{\mu} k^{\nu} \ell_a + \sum_b C_{\mu\nu\alpha\beta} \, e^{\mu}{}_{a} k^{\nu} k^{\alpha} e^{\beta}{}_{b} \, \ell_b$$

Then we rewrite this equation into three equations for the divergence, shear and rotation of the light bundle

$$\frac{d\ell_a}{d\lambda} \equiv \sum_b (\, \theta \, \delta_{ab} + \sigma_{ab} + \omega_{ab}) \ell_b$$

Since vector $\vec{\zeta}$ is perpendicular to the vector $\vec{k}$, we can take $\vec{\zeta} = \frac{\partial}{\partial \chi}$ for some parameter χ, and vector $\vec{\zeta}$ and vector $\vec{k}$ are commute each other

$$[\zeta, \ k\ ] = 0 \ \Rightarrow \ \zeta^{\alpha} \nabla_{\alpha} k^{\mu} = k^{\alpha} \nabla_{\alpha} \zeta^{\mu} = \sum \frac{d\ell_a}{d\lambda} \ e^{\mu}{}_a$$

$$\nabla_{\alpha} k^{\mu} \sum \ell_a \ e^{\alpha}{}_a = \sum \frac{d\ell_a}{d\lambda} \ e^{\mu}{}_a$$

$$\Rightarrow \qquad \frac{d\ell_a}{d\lambda} = (\nabla_{\alpha} k^{\mu}) \ e_{\mu a} \ e^{\alpha}{}_b \ \ell_b$$

$$\theta \, \delta_{ab} + \sigma_{ab} + \omega_{ab} = (\nabla_{\alpha} k_{\mu}) \ e^{\mu}{}_a \ e^{\alpha}{}_b$$

$$2\theta = \nabla^{\mu} k_{\mu} : convergence$$

$$\sigma_{ab} = \nabla_{(\nu} k_{\mu)} \ e^{\mu}{}_a e^{\nu}{}_b - \frac{1}{2} (\nabla^{\mu} k_{\mu}) \delta_{ab} : shear$$

$$\omega_{ab} = \nabla_{[\nu} k_{\mu]} \ e^{\mu}{}_a e^{\nu}{}_b : rotation$$

Derivation of the Optical Scalar Equation

$$\frac{d^2\ell_a}{d\lambda^2} \equiv \sum_b (\ \theta\,\delta_{ab} + \sigma_{ab} + \omega_{ab})\frac{d\ell_b}{d\lambda} + \sum_b (\ \frac{d\theta}{d\lambda}\ \ \delta_{ab} + \frac{d\sigma_{ab}}{d\lambda} + \frac{d\omega_{ab}}{d\lambda})\ell_b$$

$$= \sum_{b,c} (\theta\,\delta_{ab} + \sigma_{ab} + \omega_{ab})(\theta\,\delta_{bc} + \sigma_{bc} + \omega_{bc})\ell_c + \sum_b (\ \frac{d\theta}{d\lambda}\delta_{ab} + \frac{d\sigma_{ab}}{d\lambda} + \frac{d\omega_{ab}}{d\lambda})\ell_b$$

$$= \sum_b \left[(\ \theta^2\ \ + \sigma^2 - \omega^2\ \ + \frac{d\theta}{d\lambda}\ )\ \delta_{ab} + \frac{d\sigma_{ab}}{d\lambda} + 2\theta\sigma_{ab} + \frac{d\omega_{ab}}{d\lambda} + 2\theta\omega_{ab} \right] \ell_b$$

where we used

$$\sigma_{ab} = \begin{pmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & -\sigma_1 \end{pmatrix}, \quad \omega_{ab} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$$

$$\sigma_{ab}\sigma_{bc} = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} = \sigma^2\delta_{ac}, \quad \sigma^2 = (\sigma_1)^2 + (\sigma_2)^2$$

$$\omega_{ab}\omega_{bc} = \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix} = -\omega^2\delta_{ac}$$

$$(\sigma\omega)_{ac} = -(\omega\sigma)_{ac}$$

Compare this expression with

$$\frac{d^2\ell_a}{d\lambda^2} = -\frac{1}{2} R_{\mu\nu} k^\mu k^\nu \ell_a + \sum_b C_{\mu\nu\alpha\beta} \, e^\mu{}_a k^\nu k^\alpha e^\beta{}_b \, \ell_b$$

We obtain the following

$$\frac{d\theta}{d\lambda} + \theta^2 + \sigma^2 - \omega^2 = -\frac{1}{2} R_{\mu\nu} k^\mu k^\nu$$

$$\frac{d\sigma_{ab}}{d\lambda} + 2\theta\sigma_{ab} = C_{\mu\nu\alpha\beta} e^\mu{}_a k^\nu k^\alpha e^\beta{}_b$$

$$\frac{d\omega_{ab}}{d\lambda} + 2\theta\omega_{ab} = 0$$

Since the equation for rotation is homogeneous, we can assume that the rotation vanishes identically as far as it vanishes initially

Einstein Equation

Newtonian gravity can be described by Newtonian potential Φ

which satisfies Poisson equation with the mass as the source

$$\Delta\phi = 4\pi\, G\rho$$ Mass density

In Newtonian limit we know

$$g_{00} = -1 + 2\phi$$

Thus the l.h.s of Poisson equation is the Newtonian limit of the 00 component of some 2nd rank differential equation for the metric tensor

⇒ The mass density, rather the energy density .ρ in the r. h.s of Poisson equation should also the 00 component of some 2nd rank tensor (stress energy tensor/energy-momentum tensor).

$$T_{00} = \rho$$

General form of stress energy tensor for perfect fluid

$$T_{\alpha\beta} = \big(\rho + P(\rho)\big) u_\alpha u_\beta + P g_{\alpha\beta} = \rho\, u_\alpha u_\beta + P h_{\alpha\beta},$$

$$h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta :$$ 3 metric in hypersurface orthogonal to 4-velocity u

Find a 2nd rank tensor equation

$$G_{\alpha\beta} = kT_{\alpha\beta}$$

which reduces to Newtonian Poisson equation in Newtonian limit.

- $G_{\alpha\beta}$ contains 2nd order derivative for the metric tensor
- Stress energy tensor satisfies the conservation law.

$$\nabla^{\alpha} T_{\alpha\beta} = 0$$

- $G_{\alpha\beta}$ also satisfies

$$\nabla^{\alpha} G_{\alpha\beta} = 0$$

These requirements are satisfied by the following choice.

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

Thus we arrive at Einstein equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\alpha\beta}$$

Detailed calculation of Newtonian limit

The metric tensor is close to the flat Minkowski metric in Newtonian limit

$$g = \eta + h \quad \text{with} \ |h| << 1, \quad h_{00} = -2\phi$$

We use the following form of Einstein Eqn.

$$R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$$

Riemann tensor may be approximated as

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(h_{\alpha\nu,\beta\mu} - h_{\alpha\mu,\beta\nu} + h_{\beta\mu,\alpha\nu} - h_{\beta\nu,\alpha\mu}) + O(h^2)$$

$$\Rightarrow \quad R_{\mu\nu} = \frac{1}{2}\left(h^{\alpha}{}_{\nu,\alpha\mu} - h_{,\alpha\nu} + h^{\alpha}{}_{\mu,\alpha\nu} - h_{\beta\nu,\alpha}{}^{\alpha} \right)$$

where we regard h as a tensor field on the flat Minkowski spacetime

$$h^{\alpha}{}_{\mu} \equiv \eta^{\alpha\beta} h_{\beta\mu}, \qquad h \equiv \eta^{\alpha\beta} h_{\alpha\beta}$$

Thus the 00component of the Einstein equation is

$$R_{00} = \frac{1}{2}\left(2h^{\alpha}{}_{0,\alpha 0} - h_{,00} - h_{00,\alpha}{}^{\alpha} \right) \approx -\frac{1}{2} h_{00,i}{}^{,i} = \Delta\phi$$

where we used the fact that the time derivative may be neglected compared with the derivative in Newtonian limit.

The r.h.s of Einstein equation

$$T_{00} \approx \rho = O(h)$$

$$T = g^{\alpha\beta} T_{\alpha\beta} = (\eta^{\alpha\beta} - h^{\alpha\beta}) T_{\alpha\beta} = -T_{00} + T^{i}{}_{i} + O(h^2) = -T_{00} + O(h^2, hv^2)$$

$$\Rightarrow \quad T_{00} - \frac{1}{2} g_{00}\, T \approx \frac{1}{2} \rho$$

Thus the 00 component of Einstein equation becomes to

$$R_{00} = \kappa \left(T_{00} - \frac{1}{2} g_{00}\, T \right) \quad \Rightarrow \quad \Delta\phi = \frac{1}{2} \kappa\, \rho$$

Compare this equation with Poisson equation $\Delta\phi = 4\pi G\rho$, we arrive at

$$\kappa = 8\pi\, G$$

Thus we have the required equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\alpha\beta} = 8\pi\, G\, T_{\mu\nu}$$

Existence of Vacuum Solutions

Vacuum Einstein equation

$$G_{\mu\nu} = 0 \Leftrightarrow R_{\mu\nu} = 0$$

This equation does not necessarily indicate that the spacetime is flat because only 10 components out of 20 components of Riemann tensor vanishes

Example of vacuum solution

- Gravitational Wave solution
- Black hole solutions

In 3D spacetime the independent number of Riemann and Ricci tensor is both 6. Thus the vacuum is flat in 3D spacetime.

Cosmological Constant

One can add the cosmological term in Einstein equation without contradicting the conservation law

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi\, G\, T_{\mu\nu}$$

$$G_{\mu\nu} = 8\pi\, G \left(T_{\mu\nu} - \frac{\Lambda}{8\pi G} g_{\mu\nu} \right) \equiv 8\pi\, G \left(T_{\mu\nu} + T_{(\Lambda)\mu\nu} \right)$$

$\Rightarrow$ Vacuum energy

$$T_{(vac)\mu\nu} \equiv -\frac{\Lambda}{8\pi G} g_{\mu\nu}$$

$$T_{(\Lambda)\mu\nu} = (\rho_\Lambda + P_\Lambda) u_\mu u_\nu + P_\Lambda g_{\mu\nu}$$

$$\Rightarrow \quad \rho_{vac} = \frac{\Lambda}{8\pi G}, \qquad P_{vac} = -\frac{\Lambda}{8\pi G}$$

Gravitational Field around nearly static suorce

The spacetime is nearly flat

$$g = \eta + h, \quad with \;\; |h_{\mu\nu}| << 1$$

Neglecting the second order in h

$$R_{\alpha\beta\mu\nu} = \underbrace{\frac{1}{2}(h_{\alpha\nu,\beta\mu} - h_{\alpha\mu,\beta\nu} + h_{\beta\mu,\alpha\nu} - h_{\beta\nu,\alpha\mu})}_{R^L_{\alpha\beta\mu\nu}} + O(h^2)$$

$$R^L_{\alpha\beta} \equiv \eta^{\mu\nu} R^L_{\mu\alpha\nu\beta}$$

$$R^L \equiv \eta^{\alpha\beta} R^L_{\alpha\beta}$$

$$G^L{}_{\mu\nu} \equiv R^L{}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R^L$$

$$= \frac{1}{2}(h_{\alpha\nu,\beta\mu} - h_{\alpha\mu,\beta\nu} + h_{\beta\mu,\alpha\nu} - h_{\beta\nu,\alpha\mu}) - \frac{1}{2}\eta_{\mu\nu}(h_{\alpha\beta}{}^{,\alpha\beta} - h^{,\alpha}{}_{,\alpha})$$

Harmonic gauge(Lorentz gauge)

$$\bar{h}^{\mu\nu}{}_{,\nu} = 0$$

Lineraized Einstein equation in harmonic gauge

$$\partial^\alpha \partial_\alpha \bar{h}^{\mu\nu} = -16\pi\, G\, T^{\mu\nu}$$

$$\left(\partial^\alpha \partial_\alpha = \eta^{\alpha\beta} \partial_\alpha \partial_\beta = -\frac{\partial^2}{\partial t^2} + \Delta \right)$$

Analogy with EM

Maxwell Eqn.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$F^{\mu\nu}{}_{,\nu} = 4\pi\ j^\mu$$

$$A^\mu{}_{,\mu} = 0 \Rightarrow \partial^\nu \partial_\nu A^\mu = -4\pi\ j^\mu$$

Note; Harmonic?

Harmonic condition for coordinate

$$g^{\mu\nu}\nabla_\mu\nabla_\nu x^\alpha = 0 \Rightarrow \partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu x^\alpha\right) = 0 \Rightarrow \partial_\mu\left(\sqrt{-g}\,g^{\mu\alpha}\right) = 0$$

$$\boxed{\bar{g}^{\mu\nu} \equiv \sqrt{-g}\,g^{\mu\alpha}}$$

Landau-Lifshitz variable

$$\bar{g}^{\mu\nu} \equiv \eta^{\mu\nu} - \bar{h}^{\mu\nu} \quad \Rightarrow \quad \partial_\mu \bar{h}^{\mu\alpha} = 0$$

$$\bar{g}^{\mu\nu} \equiv \sqrt{-g}\,g^{\mu\alpha} \quad \Rightarrow \quad (-\bar{g})^{-1} = (-g)^2(-g)^{-1} = -g$$

$$\det A = \exp(\mathrm{Tr}\log A)$$

$$\bar{g}^{-1} = \det(\eta - \bar{h}) = \det\eta(I - \eta^{-1}\bar{h}) = -\det(I - \eta^{-1}\bar{h})$$

$$\det(I - \eta^{-1}\bar{h}) = \exp(\mathrm{Tr}\log(I - \eta^{-1}\bar{h})) \approx \exp(-\mathrm{Tr}\,\eta^{-1}\bar{h}) \approx 1 - \bar{h}_\alpha^\alpha$$

$$-\bar{g} = [\det(I - \eta^{-1}\bar{h})]^{-1} \approx 1 + \bar{h},\ \ \bar{h} \equiv \bar{h}_\alpha^\alpha$$

$$g^{\mu\alpha} = (-\bar{g})^{1/2}\bar{g}^{\mu\nu} \approx (1 + \frac{1}{2}\bar{h})(\eta^{\mu\nu} - \bar{h}^{\mu\nu}) = \eta^{\mu\nu} - \underbrace{(\bar{h}^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\bar{h})}$$

For nearly static source we can neglect the time dependence

$$\Delta\, \bar{h}^{\mu\nu} = -16\pi\, G\, T^{\mu\nu}$$

The solution which goes to 0 at spatial infinity

$$\bar{h}^{\mu\nu}(\vec{x}) = 4G\int d^3y\, \frac{T^{\mu\nu}(\vec{y})}{|\vec{x}-\vec{y}|}$$

Far field

$$\bar{h}^{\mu\nu}(\vec{x}) \approx \frac{4G}{|\vec{x}|}\int d^3y\; T^{\mu\nu}(\vec{y}) = \begin{cases} \dfrac{4GM}{r} & M \equiv \int d^3y\; T^{00}(\vec{y}) \\ \dfrac{4GP^i}{r} & P^i \equiv \int d^3y\; T^{0i}(\vec{y}) \\ \dfrac{4GZ^{ij}}{r} & Z^{ij} \equiv \int d^3y\; T^{ij}(\vec{y}) \end{cases}$$

Matter Quadruple moment

$$I^{ij} \equiv \int d^3y\; y^i y^j\; T^{00}(\vec{y})$$

$T^{\mu\nu}{}_{,\nu} = 0$ linearized theory

$$\dot{I}^{ij} = \int d^3y\; y^i y^j\; T^{00}{}_{,0} = -\int d^3y\; y^i y^j\; T^{0k}{}_{,k} = -2\int d^3y\; y^{(i}T^{j)0} = 0$$

$$\ddot{I}^{ij} = -2\int d^3y\; y^{(i}T^{j)0}{}_{,0} = 2\int d^3y\; y^{(i}T^{j)k}{}_{,k} = 2\int d^3y\; T^{ij} = Z^{ij} = 0$$

We can always chose the spatial coordinate in which the center of mass is the origin

$$D^i \equiv \int d^3y\; y^i\; T^{00}(\vec{y}) = 0$$

$$\dot{D}^i \equiv \int d^3y\; y^i\; T^{00}{}_{,0} = -\int d^3y\; y^i\; T^{0k}{}_{,\mathrm{k}} = \int d^3y\; T^{0i} = 0$$

$$P^i \equiv \int d^3y\; T^{0i} = 0$$

$$\Rightarrow \quad \bar{h}^{00} \approx \frac{4GM}{r} + O\left(\frac{1}{r^2}\right), \quad \bar{h}^{0i} = \bar{h}^{ij} = O\left(\frac{1}{r^2}\right)$$

$$h^{00} = \bar{h}^{00} - \frac{1}{2}\eta^{00}\bar{h} = \frac{2GM}{r},$$

$$h^{0i} = 0,$$

$$h^{ij} = \bar{h}^{ij} - \frac{1}{2}\delta^{ij}\bar{h} = \frac{2GM}{r}\delta^{ij}$$

Finally we have the gravitational field far from a nearly statoc source

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 + \frac{2GM}{r}\right) (dx^2 + dy^2 + dz^2)$$

or

$$ds^2 = -(1 + 2\Phi)\ dt^2 + (1 - 2\Phi)\ (dx^2 + dy^2 + dz^2)$$

$$\Phi(\vec{x}) = -G\int d^3y \frac{\rho(\vec{y})}{|\vec{x} - \vec{y}|}$$