Homogeneous & Isotropic Cosmology

- RW geometry
- Redshift
- Expansion law
- Distance-redshift relation

RW geometry

- Redshift galaxy survey shows that the spatial distribution of galaxies are homogeneous over 100 Mpc
- Temperature fluctuation in CMB is isotropic

⇒ 3D space is a constant curvature space

Theorem:

Riemann tensor of 3-dimensional constant curvature space may be written as follows

$$R_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk}),$$ with K a constant

Constant Curvature Space

3D Riemann Tensor

$$R_{AB} \equiv R_{ijkl} : A = (ij), B = (kl)$$

This can be regarded as a symmetric mapping in 3D vector space

$$Riem : \lambda_A \to R_A^B \lambda_B$$

If space is isotropic, then there will be no special direction .
This means that three eigenvalues of the mapping are the same . Otherwise, the special eigenvalue and the corresponding eigenvector defines the special direction

$$\lambda_i = \varepsilon_{ijk} \lambda^{jk}$$

This contradict the isotropy of space.

This mapping should be proportional to the identity mapping.

$$R_B^A \propto \delta_B^A$$

$$R^{ab}{}_{cd} = 2K \ \delta_c^{[a} \delta_d^{b]} = K(\delta_c^a \delta_d^b - \delta_c^b \delta_d^a)$$

Bianchi identity

$$R_{abcd;e} + R_{acde;b} + R_{adbc;a} = 0$$

From this

$$\nabla_\alpha K = 0 \quad \Rightarrow \quad \text{K=const}$$

.

Ricci tensor and Ricci scalar take the following form

$$R_{bd} = g^{ac} R_{abcd} = K(3g_{bd} - g_{bd}) = 2Kg_{bd}$$

$$R = g^{bd} R_{bd} = 6K$$

Constant curvature space is classified into 3 types depending on the signature of K

$K > 0:$ 3-sphere

$K = 0:$ Flat space

$K < 0:$ 3-d hyperboloid

Explicite form of the line element

Since space is isotropic one can foliate space by 2D sphere

$$d\ell^2 = e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Then the Ricci tensor of this metric is calculated as

$$R_{rr} = \frac{1}{r}\frac{d\lambda}{dr}, \quad R_{\theta\theta} = 1 + \frac{1}{2}re^{-\lambda}\frac{d\lambda}{dr}$$

Substitute this expression into

$$R_{bd} = 2Kg_{bd}$$

gives the following equations

$$\lambda'/r = 2Kg_{rr} = 2Ke^{\lambda}$$

$$1 + \frac{1}{2}re^{-\lambda}\lambda' - e^{-\lambda} = 2g_{\theta\theta} = 2Kr^2$$

$$\Longrightarrow \quad e^{-\lambda} = 1 - Kr^2$$

Finally we have the following line element for constant curvature space

$$d\ell^2 = \frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

RW line element

We take a comoving spatial coordinate (x^i), i=1,2,3

This means each galaxy has a constant spatial position

We take the proper time of galaxy as a cosmic time t

$$d\tau^2 = -ds^2 = -g_{00}dt^2 \quad \Rightarrow \quad g_{00} = -1$$

Each galaxy is freely falling

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \rightarrow \Gamma^i_{00} = 0 \quad \Rightarrow \quad \Gamma^i_{00} = g^{ij}g_{j0,0} = 0 \quad \Rightarrow \quad g_{i0} \equiv 0$$

Thus 4D line element may be written as

$$ds_{FRW}{}^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right]$$

$$= -dt^2 + a^2(t)\ [\ d\chi^2 + r^2(\chi)\ \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\ ]$$

where

$$\chi(r) = \int\frac{dr}{\sqrt{1-Kr^2}} = \begin{cases} \sin^{-1} r \ \ (K=+1) & \text{Closed universe} \\ r \ \ (K=0) & \text{Flat universe} \\ \sinh^{-1} r \ \ (K=-1) & \text{Open universe} \end{cases}$$

Redshift

The energy of a photon k observed by an observer u

$$E = -\vec{k} \cdot \vec{u}$$

The ratio of energy emitted at $\lambda \downarrow S$ and observed at $\lambda \downarrow O$ is

$$1 + z = \frac{(\vec{U} \cdot \vec{k})(\lambda_S)}{(\vec{U} \cdot \vec{k})(\lambda_O)}$$

Consider the rest frame of the observer and he received a photon emitted from a galaxy with radial 3 velocity $V = a d\chi / dt$

$$\vec{U}_{obs} = (1,0,0,0)$$

$$\vec{U}_{em} = \gamma\left(1,\ V / a(\lambda_S), 0, 0\right) \approx \left(1,\ V / a(\lambda_S), 0, 0\right)$$

From isotropy of space, the observer is located at the origin of the spatial coordinate so that a photon propagate radially

$$\vec{k} = \left(k^0, k^0 / a\right)$$

$$0 = \frac{dk^0}{d\lambda} + \Gamma^0_{\chi\chi}\left(k^\chi\right)^2 = k^0 \frac{dk^0}{dt} + \frac{\dot{a}}{a}\left(k^0\right)^2 = 0 \Rightarrow k^0 \propto \frac{1}{a}$$

$$E_{obs} = h\nu_{obs} = -\vec{k}\left(\lambda_{obs}\right)\cdot\vec{U}_{obs} = k^0_{obs}$$

$$E_{em} = h\nu_{em} = -\vec{k}\left(\lambda_{em}\right)\cdot\vec{U}_{em} = k^0_{em}(1-V)$$

$$\frac{1}{1+z} = \frac{\nu_{obs}}{\nu_{em}} = \frac{k^0_{obs}}{k^0_{em}(1-V)} = \frac{a_{em}}{a_{obs}(1-V)}$$

No peculiar motion in an exactly homogeneous and isotropic universe

$$1+z = \frac{a_0}{a(t_{em})} = \frac{1}{a(t_{em})}$$

Dynamics of Universe

$$ds_{FRW}{}^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right]$$

+ Einstein equation

Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0$$

We have 3 kinds of energy

$$\rho = \rho_m + \rho_r + \rho_\Lambda$$

with $P_m = 0$, $P_r = 1/3\, \rho_r$, $P_\Lambda = -\rho_\Lambda$

Friedmann equation in terms of density parameter

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3}(\rho_m + \rho_r + \rho_\Lambda)$$

Using the density parameters

$$\Omega_X(a) \equiv \frac{\rho_X(a)}{\rho_{cr}(a)}, \quad \rho_{cr}(a) \equiv \frac{3H^2}{8\pi G}$$

We have

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_m + \rho_r + \rho_\Lambda) - \frac{K}{a^2}$$

$$= H^2\left(\Omega_m(a) + \Omega_r(a) + \Omega_\Lambda(a) - \frac{K}{a^2 H^2}\right) \Leftarrow \Omega_X \equiv \frac{\rho_X(a)}{\rho_{cr}(a)}$$

$$\Rightarrow \quad \frac{K}{a^2 H^2} = \Omega_m(a) + \Omega_r(a) + \Omega_\Lambda(a) - 1 = \Omega_{tot}(a) - 1$$

Thus Universe is flat when the total density parameter is equal to 1

Friedmann equation in terms of the present density parameters

$$\Omega_{X,0} \equiv \frac{\rho_{X,0}}{\rho_{cr,0}}, \qquad \rho_{cr,0} = \frac{3H_0^2}{8\pi G}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\frac{\rho_{m0}}{a^3} + \frac{\rho_{r0}}{a^4} + \rho_\Lambda\right) - \frac{K}{a^2}$$

$$\Rightarrow H^2 = H_0^2\left(\frac{\rho_{m0}}{\rho_{cr,0}}\frac{1}{a^3} + \frac{\rho_{r0}}{\rho_{cr,0}}\frac{1}{a^4} + \frac{\rho_\Lambda}{\rho_{cr,0}} - \frac{K}{a^2 H_0^2}\right)$$

$$\Rightarrow H^2 = H_0^2\left(\frac{\Omega_{m0}}{a^3} + \frac{\Omega_{r0}}{a^4} + \Omega_{\Lambda 0} - \frac{K}{H_0^2}\right)$$

Hubble parameter as a function of redshift

$$\lambda_{obs} = \lambda_{em}(1+z) \Rightarrow 1+z = \frac{1}{a(t)}$$

$$H^2(z) = H_0^2\left(\Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{\Lambda 0}\right)$$

Evolution of density parameters with totally flat case(Λ=0)

$$\Omega_X(a) \equiv \frac{\rho_X(a)}{\rho_{cr}(a)}$$

$$\rho_{cr}(a) \equiv \frac{H^2}{8\pi G} = \frac{H_0^2}{8\pi G}\left(\frac{\Omega_{m0}}{a^3} + \frac{\Omega_{r0}}{a^4} + \Omega_{\Lambda 0}\right)$$
$$= \rho_{cr,0}\left(\frac{\Omega_{m0}}{a^3} + \frac{\Omega_{r0}}{a^4} + \Omega_{\Lambda 0}\right)$$

$$\Omega_m(a) \equiv \frac{\rho_m(a)}{\rho_{cr}(a)} = \frac{\rho_{m,0}a^{-3}}{\rho_{cr,0}\left(\Omega_{m,0}a^{-3} + \Omega_{r,0}a^{-4} + \Omega_{\Lambda,0}\right)}$$
$$= \frac{\Omega_{m,0}(1+z)^3}{\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\Lambda,0}}$$

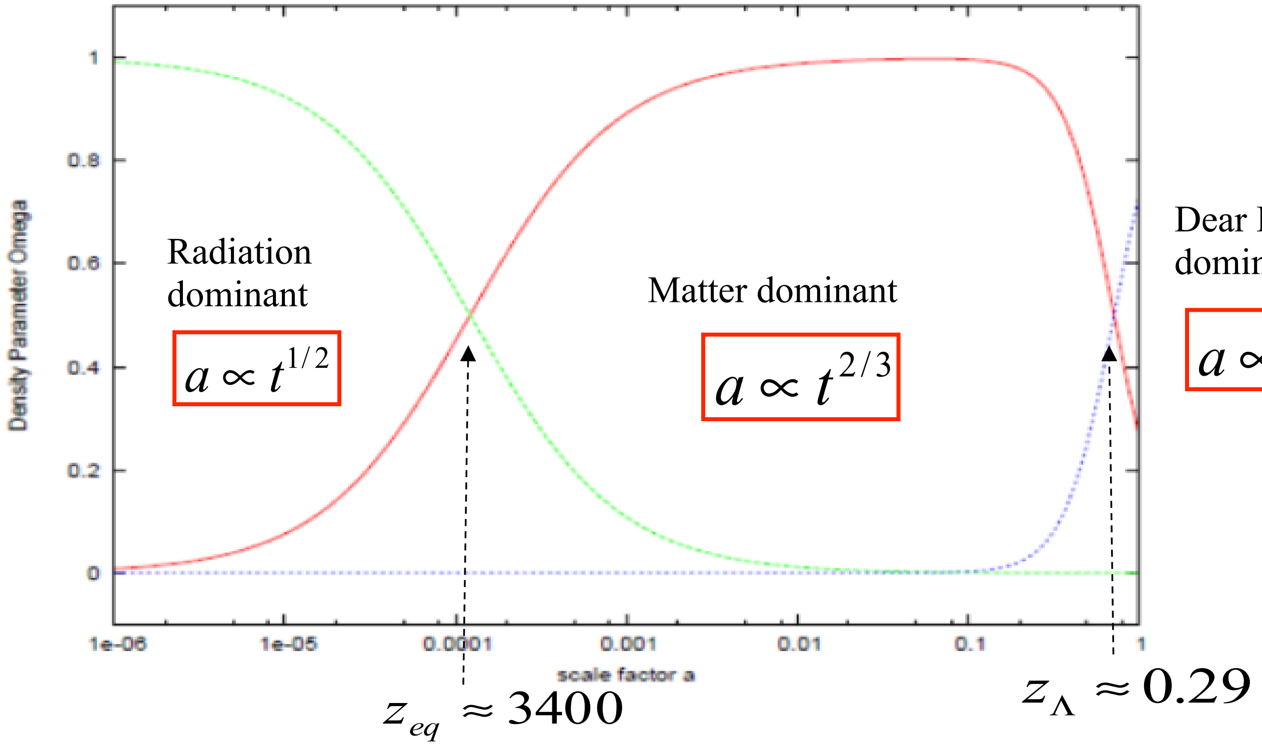
In the following we assume totally flat universe K=0

$$\Omega_r(z) = \frac{\Omega_{r,0}(1+z)^4}{\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0}}$$

$$\Omega_m(z) = \frac{\Omega_{m,0}(1+z)^3}{\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0}}$$

$$\Omega_\Lambda(z) = \frac{\Omega_{\Lambda,0}}{\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0}}$$

These relations are obtained by noticing $\rho_{cr} = \frac{3H^2}{8\pi G} = \rho_{cr,0}\left(\Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{\Lambda 0}\right)$



Dear Energy dominant

$a \propto e^t$

$$1 + z_{eq} = \frac{1}{a_{eq}} = \frac{\Omega_{m0}}{\Omega_{r0}}$$

$$1 + z_\Lambda = \frac{1}{a_\Lambda} = \left(\frac{\Omega_{\Lambda 0}}{\Omega_{m0}}\right)^{1/3} \approx 1.29$$

Important epoch in totally flat universe

$$H^2(z) = H_0^2\left(\Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{\Lambda 0}\right)$$

Radiation-matter equality time t_{eq} : $\rho_r(t_{eq}) = \rho_m(t_{eq})$

$$\rho_r(t) = \rho_{r0}a^{-4}(t) = \rho_{r0}(1+z)^4$$

$$\rho_{r0} = \frac{\pi^2 k^4}{30\hbar^3} g_* T_0^4 \approx 7.80\times10^{-31} c^2 \ kg/m^3 \Leftarrow T_0 \approx 2.725K$$

$$\rho_m(t) = \rho_{m0}a^{-3}(t) = \rho_{m0}(1+z)^3 = \Omega_{m,0}\rho_{cr,0}(1+z)^3$$

$$\rho_{cr,0} \approx 1.88\times10^{-29} h^2 \text{ g/cm}^3$$

$$1+z_{eq} = \frac{\rho_{m0}}{\rho_{r0}} = \frac{\Omega_{m,0}}{\Omega_{r,0}} \approx 2.41\times10^4(\Omega_{m0}h^2) \approx 3400$$

Matter-cosmological constant equality time t_Λ : $\rho_m(t_\Lambda) = \rho_\Lambda(t_\Lambda)$

$$\Omega_{m,0}\rho_{cr,0}(1+z_\Lambda)^3 = \rho_\Lambda \Rightarrow z_\Lambda = \left(\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)^{1/3} - 1 \approx 0.29$$

Expansion Behavior in Transition from MD to Λ Dominant

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\frac{\Omega_{m0}}{a^3} + \Omega_{\Lambda 0}\right)$$

$$H_0 t = \int_0^a \frac{\sqrt{a}\,da}{\sqrt{\Omega_{m,0} + \Omega_{\Lambda,0} a^3}}$$

Since $\displaystyle \frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{x^2+1}} \Rightarrow \frac{d}{dx}\sinh^{-1} x^{3/2} = \frac{3}{2}\sqrt{\frac{x}{x^3+1}}$

$$a(t) = \left(\frac{\Omega_{m,0}}{1-\Omega_{m,0}}\right)^{1/3} \sinh^{2/3}\left(\frac{3}{2}\sqrt{1-\Omega_{m,0}}\,H_0 t\right) = a_\Lambda \sinh^{2/3}\left(\frac{3}{2}\sqrt{1-\Omega_{m,0}}\,H_0 t\right)$$

We used

$$1 + z_\Lambda = \frac{1}{a_\Lambda} = \left(\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)^{1/3} = \left(\frac{1-\Omega_{m,0}}{\Omega_{m,0}}\right)^{1/3}$$

Since $\cosh x + \sinh x = e^x$

$$t(a) = \frac{2}{3H_0\sqrt{1-\Omega_{m,0}}} \ln\left[\left(\frac{a}{a_\Lambda}\right)^{3/2} + \sqrt{1+\left(\frac{a}{a_\Lambda}\right)^3}\right]$$

For a=1(at present)

$$t_0 = \frac{2}{3H_0\sqrt{1-\Omega_{m,0}}} \ln\left[\frac{1+\sqrt{1-\Omega_{m,0}}}{\sqrt{\Omega_{m,0}}}\right] \simeq 13.8\,\mathrm{Gyr}$$

Standard Model

Current standard model is a totally flat universe with the following parameters

$$K = 0 \qquad \Omega_{\Lambda 0} \approx 0.69,\ \ \Omega_{m0} \approx 0.31,\ \ \Omega_{r0} \approx 8.4\times 10^{-5},$$

$$H_0 \approx 67.8\ \mathrm{km\,s^{-1}Mpc^{-1}}$$

Useful numbers

$$z_{eq} \approx 3400 \qquad t_{eq} \approx 3.4\times 10^{-6} H_0^{-1} \approx 51{,}000 yr$$

$$a_\Lambda = \left(\frac{\Omega_{m0}}{\Omega_{\Lambda 0}}\right)^{1/3} \approx 0.77, \quad z_\Lambda \approx 0.29, \quad t_\Lambda \sim 10.3 Gyr$$

$$t_0 = \frac{2}{3H_0\sqrt{1-\Omega_{m0}}}\ \ln\left[\frac{1+\sqrt{1-\Omega_{m0}}}{\sqrt{\Omega_{m0}}}\right] \sim 13.8\ Gyr$$

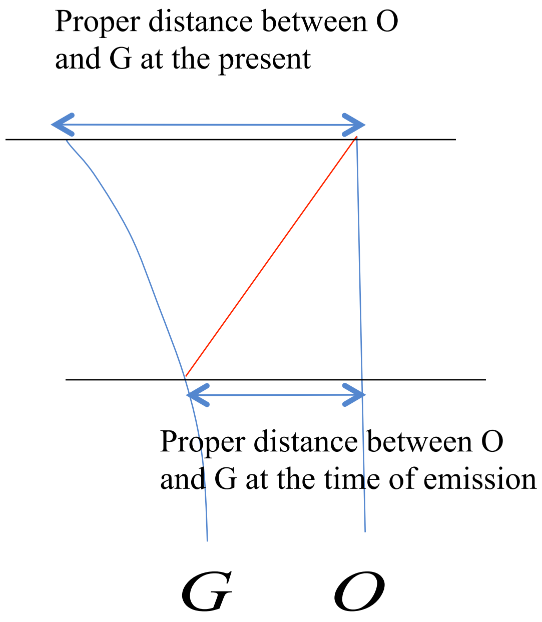
Distance-redshift relation

What is distance in cosmology?

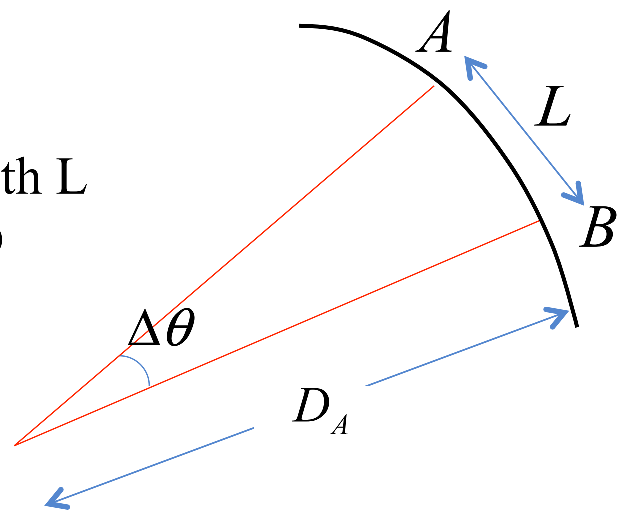
Proper distance(measure both ends at the same time) is useless because what we observe today was existed in the past

We only have operationally defined“distance” directly related with observation. Direct observable is redshift and thus we need the relation between the operationally distance and redshift

- Angular diameter distance
- Luminosity distance



Angular diameter distance



In Euclidean geometry an object with proper length L
is seen as an angular scale Δθ   from the distnce D

$$\Delta\theta = \frac{L}{D}$$

We use this relation as a definition of distance. Namely if an object whose
proper distance L is known(standard ruler) is seen as an angular scale Δθ,
then we define the distance to the object as

$$D_A \equiv \frac{L}{\Delta\theta}$$

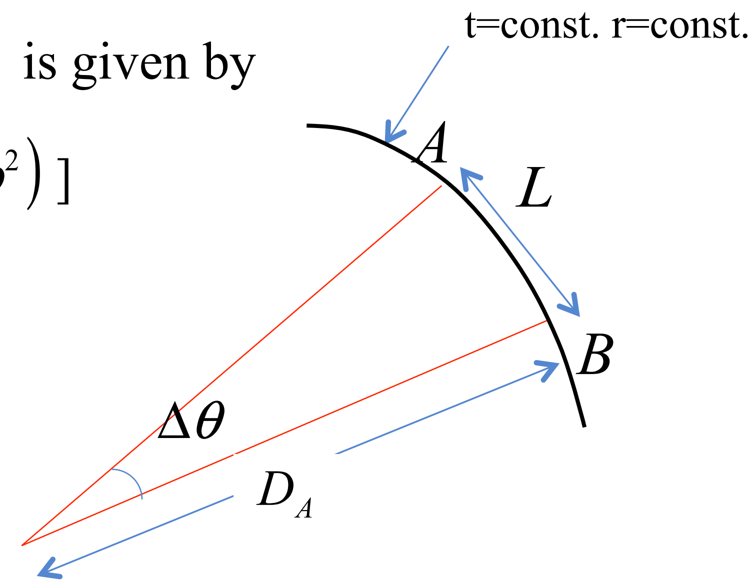
This is angular distance.
The problem of this distance is that there is no source whose proper
distance is known.

Angular diameter distance as a function of redshift

Consider two nearby events A and B on the same time and radial coordinates, and also same angular coordinate φ ＝const.

Since the distance between arbitrary two events is given by

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + r^2(\chi)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)]$$



The proper distance between event A and B

$$L = [-ds^2(dt = dr = d\phi = 0)]^{1/2} = a(t)\ r\,\Delta\theta$$

Thus

$$D_A(z) = a(t)\ r(z) = \frac{r(z)}{1+z}$$

Along the light path

$$\int_0^r \frac{dr}{\sqrt{1-Kr^2}} = \int_t^{t_0} \frac{dt}{a(t)} \quad (=\chi\ )$$

$$\chi(r) = \int \frac{dr}{\sqrt{1-Kr^2}} = \begin{cases} \sin^{-1} r\ \ (K=+1) \\ r\quad (K=0) \\ \sinh^{-1} r\ \ (K=-1) \end{cases} \quad \Rightarrow \quad r(z) = \begin{cases} \sin\chi\ (z),\ \ (K=+1) \\ \chi\ (z),\quad (K=0) \\ \sinh\chi\ (z),\ \ (K=-1) \end{cases}$$

Calculation of χ(z) K=0 case

$$cdt = -a(t)d\chi$$

$$\chi(z) = \int_t^{t_0} \frac{dt}{a(t)} = \int_t^{t_0} \frac{da}{a}\frac{dt}{da} = \int_t^{t_0} \frac{da}{a^2}\frac{1}{H} = \int_0^z \frac{dz}{H(z)} = \frac{1}{H_0}\int_0^z \frac{dz}{E(z)}$$

$$E(z) = [\Omega_{m0}(1+z)^3 + \Omega_{\Lambda 0}]^{1/2}$$

$K = \Lambda = 0$ Einstein - de Sitter universe

$$\chi(z) = \frac{1}{H_0}\int_0^z \frac{dz}{E(z)} = \frac{1}{H_0}\int_0^z \frac{dz}{(1+z)^{3/2}} = \frac{2}{H_0}\left[1 - \frac{1}{(1+z)^{1/2}}\right]$$

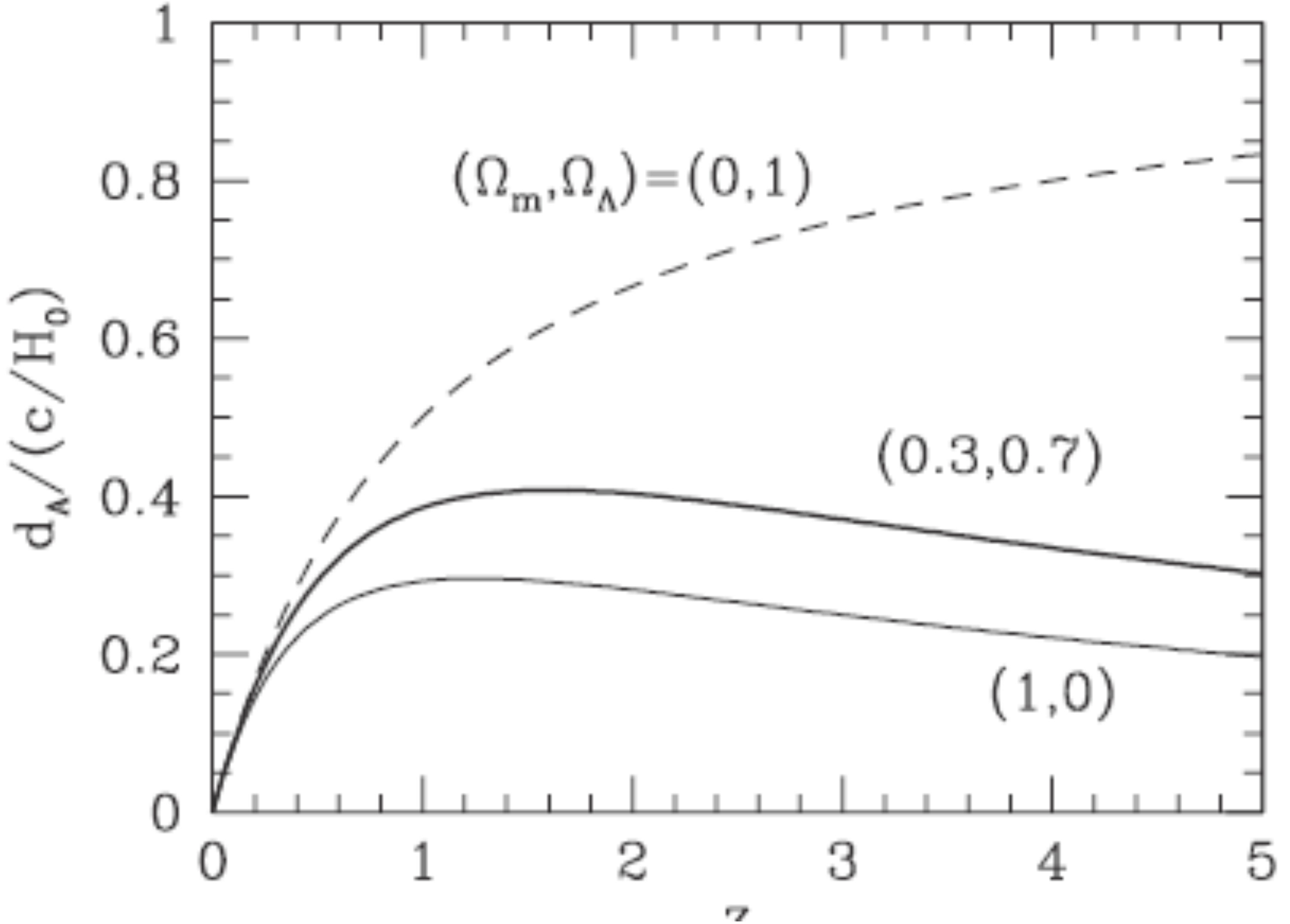
$$D_A(z) = a(z)\chi(z) = \frac{2}{H_0}\left[\frac{1}{1+z} - \frac{1}{(1+z)^{3/2}}\right]$$

In flat model K=0

$$\chi(z) = \frac{1}{H_0}\int_0^z \frac{dz}{\sqrt{\Omega_{m0}(1+z)^3 + \Omega_{\Lambda 0}}} \to \frac{2}{H_0\sqrt{\Omega_{m0}}}\left[1 - \frac{1}{\sqrt{1+z}}\right] \approx \frac{2}{H_0\sqrt{\Omega_{m0}}} \quad \text{as } z \gg 1$$

$$D_A(z) = a(z)\chi(z) = \frac{2}{H_0\sqrt{\Omega_{m0}}}\frac{1}{1+z} \quad \text{for large z}$$

Angular diameter distance



Luminosity distance

In Euclidean geometry the observed energy flux f at distance D from a source with absolute luminosity L may be written as s

$$f = \frac{L}{4\pi D^2}$$

One can use this relation as a definition of "distance"

$$D_L \equiv \left(\frac{L}{4\pi f} \right)^{1/2}$$

This is called the luminosity distance

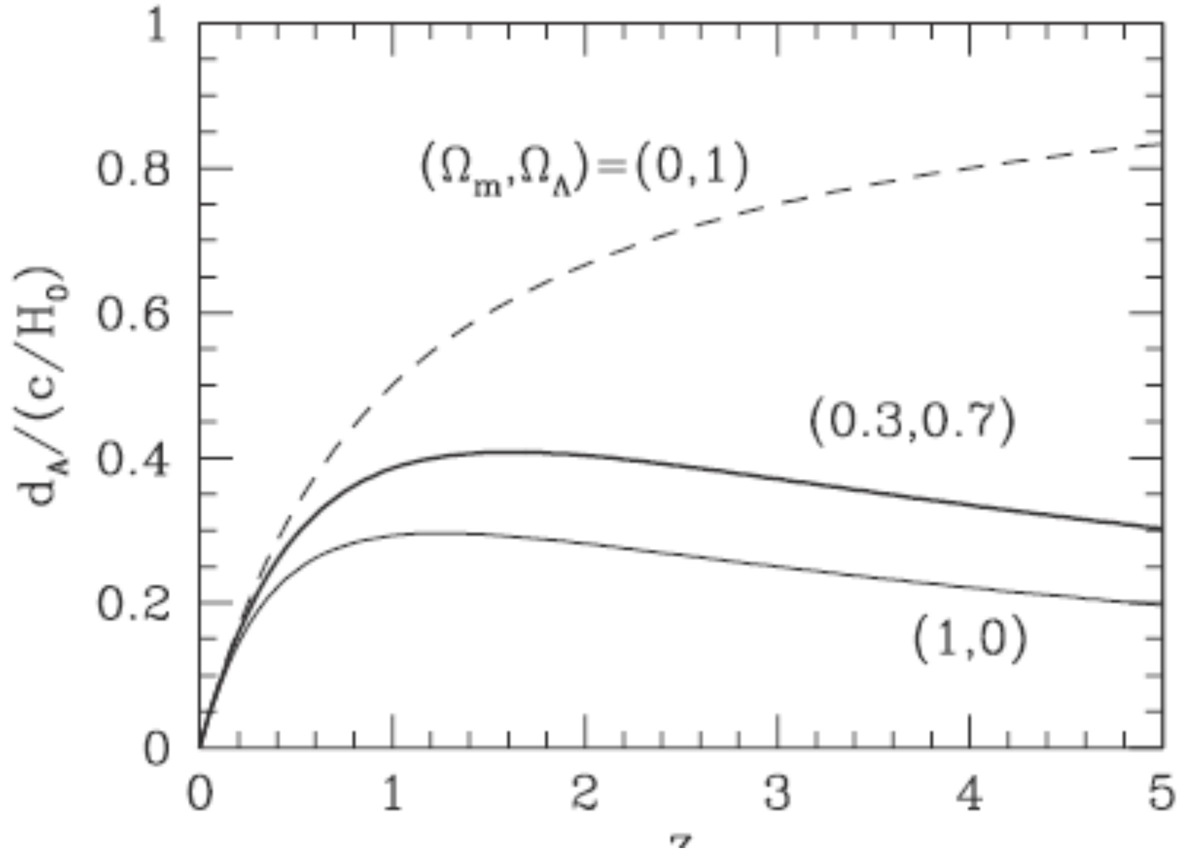
Consider a light source with a fixed comoving radial coordinate χ

$$f_\nu(\nu_{obs})\Delta\nu_{obs}\Delta t_{obs} = \frac{L_\nu(\nu_{em})\Delta\nu_{em}\Delta t_{em}}{4\pi\ r^2(\chi(z))} = \frac{L_\nu(\nu_{em})\Delta\nu_{obs}\Delta t_{obs}}{4\pi\ r^2(\chi(z))} \times \frac{1}{(1+z)^2}$$

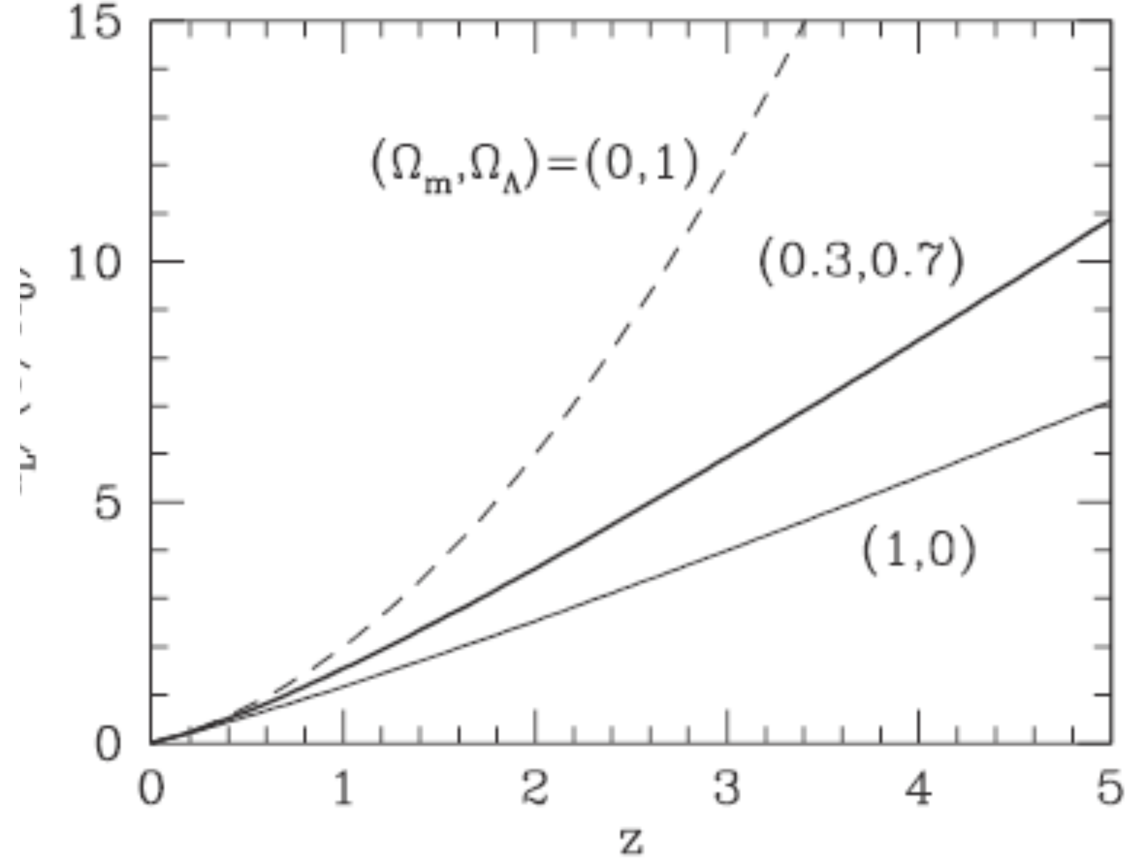
$$\Rightarrow \boxed{D_L(z) = (1+z)\ r(z)}$$

Flux is energy per unit area and unit time

Angular diameter distance and Luminosity distance



Angular diameter distance



Luminosity distance

Inhomogeneous cosmology

The line element of realistic inhomogeneous universe

$$ds^2 = -(1+2\Psi)dt^2 + a^2(t)(1-2\Psi)(dx^2+dy^2+dz^3)$$

$$\rho_m(z) = \bar{\rho}_b(1+\delta_m(z))$$

$$\Delta\Psi = 4\pi G a^2 \bar{\rho}_b \delta_m$$

Note

Although we consider only linear order in the potential Ψ, it does not mean that the density contrast $\delta\downarrow m$ is also small. It can be very large as far as the spatial scale of the contrast is much smaller than the horizon size.

$$\Delta\Psi = 4\pi G a^2 \bar{\rho}_b \delta_m \Rightarrow \frac{\Psi}{\ell^2} \sim \frac{\delta}{L_H^2} \qquad \Leftarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\bar{\rho}_b$$

$$\delta \sim \left(\frac{L_H}{\ell}\right)^2 \Psi >> 1$$

Geodesic equation in inhomogeneous universe

It is convenient to work in a conformally related spacetime

$$d\eta = \frac{dt}{a(t)}$$ Conformal time

$$ds^2 = a^2\left[-(1+2\Psi)dt^2 + (1-2\Phi)(dx^2 + dy^2 + dz^3)\right] \equiv a^2 \tilde{g}_{\mu\nu} dx^\mu dx^\nu = a^2 d\tilde{s}^2$$

More generally one can prove the following

When two metric g and g is related conformally as

$$g_{\mu\nu} = \Omega^2(x)\tilde{g}_{\mu\nu}$$

then a null geodesic in the metric g is also a null geodesic in g

$$k^2 = g_{\mu\nu}k^\mu k^\nu = 0, \quad k^\alpha \nabla_\alpha k^\mu = k^\alpha\left(\partial_\alpha k^\mu + \Gamma^\mu_{\alpha\beta}(g)k^\beta\right) = 0$$

Christoffel symbol in the metric g

$$\Gamma^{\mu}_{\alpha\beta}(\tilde{g}) = \frac{1}{2}\tilde{g}^{\mu\nu}\left(\partial_\alpha \tilde{g}_{\nu\beta} + \partial_\beta \tilde{g}_{\alpha\nu} - \partial_\nu \tilde{g}_{\alpha\beta}\right) = \Gamma^{\mu}_{\alpha\beta}(g) + \Omega^{-1} g^{\mu\nu}\left(g_{\nu\beta}\,\partial_\alpha\Omega + g_{\alpha\nu}\partial_\beta\Omega - g_{\alpha\beta}\partial_\nu\Omega\right)$$

$$k^{\alpha}\tilde{\nabla}_{\alpha}k^{\mu} = k^{\alpha}\left(\partial_\alpha k^{\mu} + \Gamma^{\mu}_{\alpha\beta}(\tilde{g})k^{\beta}\right) = k^{\alpha}\left(\partial_\alpha k^{\mu} + \Gamma^{\mu}_{\alpha\beta}(g)k^{\beta} + 2(\partial_\alpha \ln\Omega)k^{\beta}\right) = 2(\partial_\alpha \ln\Omega)k^{\alpha}k^{\beta}$$

We introduce a new parameter λ by the following condition

$$\frac{d\lambda}{d\tilde{\lambda}} = \Omega^2$$

Then the 4-momentum in this new parameter

$$\tilde{k}^{\mu} = \frac{dx^{\mu}}{d\tilde{\lambda}} = \Omega^{-2}k^{\mu}$$

$$\tilde{k}^{\alpha}\tilde{\nabla}_{\alpha}\tilde{k}^{\mu} = \tilde{k}^{\alpha}\left(\partial_\alpha \tilde{k}^{\mu} + \Gamma^{\mu}_{\alpha\beta}(\tilde{g})\tilde{k}^{\beta}\right) = \Omega^{-2}k^{\alpha}\left(\partial_\alpha k^{\mu} + \Gamma^{\mu}_{\alpha\beta}(g)k^{\beta}\right) = 0$$

Now we solve the geodesic equation in inhomogeneous universe

$$\frac{d\tilde{k}^\mu}{d\tilde{\lambda}} + \Gamma^\mu{}_{\alpha\beta}\tilde{k}^\alpha\tilde{k}^\beta = 0$$

$$\tilde{g}_{\mu\nu}\tilde{k}^\mu\tilde{k}^\nu = 0$$

Non vanishing Christoffel symbols are

$$\Gamma^0{}_{00} = \Psi', \ \Gamma^0{}_{0i} = \Psi_{,i}, \ \Gamma^0{}_{ij} = -\Phi'\delta_{ij}$$

$$\Gamma^i{}_{00} = \Psi_i, \ \Gamma^i{}_{0j} = \Phi'\delta^i_j, \ \Gamma^i{}_{jk} = -\Phi_{,j}\delta^i_k - \Phi_{,k}\delta^i_j + \Phi^{,i}\delta_{ij}$$

$$\Psi' = \partial\Psi / \partial\eta$$

$$\frac{d\tilde{k}^0}{d\tilde{\lambda}} + \Gamma^0_{00}(\tilde{k}^0)^2 + 2\Gamma^0_{0i}\tilde{k}^0\tilde{k}^i + \Gamma^0_{ij}\tilde{k}^i\tilde{k}^j = 0$$

$$\frac{d\tilde{k}^i}{d\tilde{\lambda}} + \Gamma^i_{00}(\tilde{k}^0)^2 + 2\Gamma^i_{0j}\tilde{k}^0\tilde{k}^j + \Gamma^i_{jk}\tilde{k}^j\tilde{k}^k = 0$$

In the following we omit the tilde on k

Time component of geodesic equation

Application to CMB Temperature fluctuatoon

$$\frac{dk^0}{d\lambda} + \Psi'(k^0)^2 + 2\Psi_{,i}k^0k^i - \Phi'\delta_{ij}k^ik^j = 0$$

Remembering $k^\mu = dx^\mu / d\lambda$ and $k^2 = 0$, and neglecting second order in the potential

$$\frac{dk^0}{d\eta} + k^0\left(\Psi' + 2\Psi_{,i}n^i - \Phi'\right) = 0 \qquad \vec{k} = (k^0, k^0 n^i) \text{ with } n^i n^i = 1$$

$$\frac{dk^0}{d\lambda} - (\Psi' + \Phi')k^0 + 2k^0\left(\frac{\partial}{\partial\eta} + n^i\partial_i\right)\Psi = 0$$

$$\frac{d\ln k^0}{d\lambda} = (\Psi' + \Phi') - 2\left(\frac{\partial}{\partial\eta} + n^i\partial_i\right)\Psi$$

Then integrating this equation from the last scattering surface to the present

$$\ln\frac{k^0(\eta_0)}{k^0(\eta_{LS})}=\int_{\eta_{LS}}^{\eta_0} d\eta(\Psi'+\Phi')-2\left[\Psi(\eta_0)-\Psi(\eta_{LS})\right]$$

$$\Rightarrow \quad \frac{k^0(\eta_0)-k^0(\eta_{\rm LS})}{k^0(\eta_{\rm LS})}=\int_{\eta_{\rm LS}}^{\eta_0} d\eta'\left(\Psi'+\Phi'\right)-2\left[\Psi(\eta_0)-\Psi(\eta_{\rm LS})\right]$$

The energy of a CMB photon measured by an observer $\vec{U}=U^0\left(1,v^i\right)$

$$U^2=-1\Rightarrow U^0=1-\Psi$$

$$\omega(\eta)=-g_{\mu\nu}U^\mu k^\nu=\left[1+\Psi(\eta)-n^i v^i(\eta)\right]k^0(\eta)$$

$$\frac{\omega(\vec{n},\eta_0)-\omega(\vec{n},\eta_{\rm LS})}{\omega(\vec{n},\eta_{\rm LS})}=\int_{\eta_{\rm LS}}^{\eta_0} d\eta'\left(\Psi'+\Phi'\right)-\left[\Psi(\eta_0)-\Psi(\eta_{\rm LS})+n^i\left(v^i(\eta_{\rm LS})-v^i(\eta_0)\right)\right.$$

$$\frac{\omega(\vec{n},\eta_0)-\omega(\vec{n},\eta_{\rm LS})}{\omega(\vec{n},\eta_{\rm LS})} = \int_{\eta_{\rm LS}}^{\eta_0} d\eta' \left(\Psi' + \Phi'\right) - \left[\Psi(\eta_0) - \Psi(\eta_{\rm LS}) + n^i \left(v^i(\eta_{\rm LS}) - v^i(\eta_0)\right)\right]$$

Neglecting the potential and Doppler effect today, and remembering the energy density of photon fluid is proportional to T↑4

$$\delta_\gamma = \frac{\delta\rho_\gamma}{\rho_\gamma} = 4\frac{\delta T}{T}$$

We arrive at the equation for the temperature fluctuation of CMB observed today

$$\frac{\delta T}{T} = \underbrace{\frac{1}{4}\delta\rho_\gamma(\eta_{\rm LS}) + \Psi(\eta_{\rm LS})}_{\text{Sacks-Wolfe effect}} + \underbrace{\int_{\eta_{\rm LS}}^{\eta_0} d\eta' \left(\Psi' + \Phi'\right)}_{\text{Integrated S-W}} + n^i v^i(\eta_{\rm LS})$$

Sacks-Wolfe effect

Integrated S-W

Sacks-Wolfe effect

$$\left(\frac{\delta T}{T}\right)_{SW} = \frac{1}{4}\delta\rho_\gamma + \Psi(\eta_{LS}) = \left.\frac{\delta T}{T}\right|_{LS} + \Psi(\eta_{LS})$$

One can regard the effect of the potential at last scattering surface as the shift of the time of the last scattering

$$-(1+2\Psi)dt^2 \to d(t+\delta t)^2 \Rightarrow \frac{\delta t}{t} = \Psi$$

At the last scattering the universe is matter dominant , thus $a(t) \propto t↑2/3$

$$\left.\frac{\delta T}{T}\right|_{LS} = -\frac{\delta a}{a} = -\frac{2}{3}\frac{\delta t}{t} = --\frac{2}{3}\Psi(\eta_{LS})$$

$$\left(\frac{\delta T}{T}\right)_{SW} = \left.\frac{\delta T}{T}\right|_{LS} + \Psi(\eta_{LS}) = \frac{1}{3}\Psi(\eta_{LS})$$

Higher density regions is observed as cold spots

Integrated Sachs-Wolfe effect

$$\left(\frac{\delta T}{T} \right)_{ISW} = \int_{\eta_{LS}}^{\eta_0} d\eta \left(\Psi' + \Phi' \right)$$

When the universe is exactly matter dominant the potentials are constant in time. Therefore there are two region where ISW effect becomes impotant.is

Early ISW

Effect of radiation around the last scattering surface

Late ISW

Effect of dark energy around z=0.3

Space component of geodesic equation

Application to gravitational lensing

In the following we take $\Phi=\Psi$. Then the geodesic equation becomes

$$\frac{d^2\eta}{d\lambda^2} + 2\Psi_{,i}n^i\left(\frac{d\eta}{d\lambda}\right)^2 = 0$$

$$\frac{d^2x^i}{d\lambda^2} + \left(-2\Psi' n^i + 2(\delta^{ij} - n^i n^j)\partial_j\Psi\right)\left(\frac{d\eta}{d\lambda}\right)^2 = 0$$

Combined these two equations give us the following

$$\frac{d^2x^i}{d\eta^2} - 2\frac{d\Psi}{d\eta}n^i + 2\left(\delta^{ij} - n^i n^j\right)\partial_j\Psi = 0$$

Some detail

$$\frac{d^2x^i}{d\lambda^2} = \frac{d}{d\lambda}\left(\frac{d\eta}{d\lambda}\frac{dx^i}{d\eta}\right) = \frac{d^2\eta}{d\lambda^2}\frac{dx^i}{d\eta} + \left(\frac{d\eta}{d\lambda}\right)^2\frac{d^2x^i}{d\eta^2} = \frac{d^2\eta}{d\lambda^2}\frac{k^i}{k^0} + \left(\frac{d\eta}{d\lambda}\right)^2\frac{d^2x^i}{d\eta^2}$$

Suppose that the light ray propagates toward positive z-direction so that n =(0, 0, 1)

Direction of propagation changes by the potential, but its change is small so that we neglect the effect when it is multiplied by another small quantity such as the potential

$$\frac{d^2x^a}{d\eta^2} + 2\partial_a \Psi = 0, \; a = 1, 2$$

$$\frac{d^2x^3}{d\eta^2} - 2\frac{d\Psi}{d\eta} = 0$$